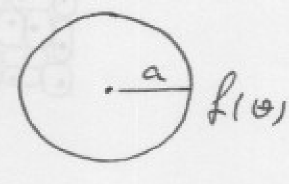
(2.5.2) Laplace equation on a circular disk

$$\begin{cases} \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 & \text{on } 0 \le r \le a \\ & -\pi \le \theta \le \pi \\ u(a,\theta) = f(\theta) & \end{cases}$$



Note: not enough side conditions!

But $0 \le r \le a$ $\quad -\pi \le \theta \le \pi$, so mathematically

we need conditions at $r=0$ $\quad r=a$, $\quad \theta = \pm\pi$.

at $r=0$: $\quad |u(0,\theta)| < \infty$ $\quad$ no singularity

at $\theta = \pm\pi$: continuity:

$$\left.\begin{aligned} u(r,-\pi) &= u(r,\pi) \\ \frac{\partial u}{\partial \theta}(r,-\pi) &= \frac{\partial u}{\partial \theta}(r,\pi) \end{aligned}\right\} \text{periodic boundary conditions}$$

Now we have enough conditions and we can use separation

1) $u(r,\theta) = \phi(\theta)\, G(r)$

2) $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)\phi + \frac{1}{r^2}\phi'' G = 0$

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\phi'' = \lambda$$

homogeneous side conditions:

$$G(r)\,\phi(-\pi) = G(r)\,\phi(\pi) \Rightarrow \phi(-\pi) = \phi(\pi)$$

$$G(r)\,\phi'(-\pi) = G(r)\,\phi'(\pi) \Rightarrow \phi'(-\pi) = \phi'(\pi)$$

$$| G(0)\, \phi(\theta) | < \infty .$$

r- problem: $\quad r \dfrac{d}{dr}\left(r \dfrac{dG}{dr}\right) = \lambda G \,, \quad |G(0)| < \infty$

θ- problem: $\quad \left.\begin{aligned} \phi'' &= -\lambda \phi \\ \phi(-\pi) &= \phi(\pi) \\ \phi'(-\pi) &= \phi'(\pi) \end{aligned}\right\}$ complete.

3) Solve θ - problem first:

<u>case 1</u> $\lambda < 0$: $\quad \phi(\theta) = a e^{\sqrt{|\lambda|}\,\theta} + b e^{-\sqrt{|\lambda|}\,\theta}$

$$\phi'(\theta) = a\sqrt{|\lambda|}\, e^{\sqrt{\ }\theta} - b\sqrt{\ }\, e^{-\sqrt{\ }\theta}$$

$\phi(-\pi) = \phi(\pi)$: $\quad a e^{-\sqrt{\ }\pi} + b e^{\sqrt{\ }\pi} = a e^{\sqrt{\ }\pi} + b e^{-\sqrt{\ }\pi}$

$$\Rightarrow \quad a = b$$

$$\phi'(-\pi) = \phi'(\pi) \Rightarrow \quad a = -b$$

together $\quad a = -a \quad \Rightarrow \quad a = 0 = b.$

Only the trivial solution.

<u>Case 2</u> $\lambda = 0 \quad \phi(\theta) = c_1 \theta + c_2$

$$\phi(-\pi) = \phi(\pi) \Rightarrow c_1 = 0$$

$\phi'(-\pi) = \phi'(\pi)$ is true.

Hence we find one solution $\phi_0(\theta) = 1$

Case 3 $\quad \lambda > 0: \quad \phi(\theta) = \alpha \cos(\sqrt{\lambda}\,\theta) + \beta \sin(\sqrt{\lambda}\,\theta)$

sine and cosine are periodic on $[\pi, -\pi]$ if

$\sqrt{\lambda}\,\pi = n\pi \Rightarrow \lambda_n = n^2$ eigenvalues

eigenfunctions $\quad \phi_n(\theta) = \alpha \cos(n\theta) + \beta \sin(n\theta)$.

Again, the case $n=0$, $\lambda_0 = 0$, $\phi_0 = 1$

can be included in case 3.

4)

$$\frac{r}{\cancel{G}} \frac{d}{dr}\left(r \frac{dG}{dr}\right) = \lambda_n = n^2 G$$

$$\parallel$$

$$r \frac{dG}{dr} + r^2 \frac{d^2G}{dr^2} = n^2 G \qquad |G(0)| < \infty.$$

Eulers equation. Solve by $G(r) = r^{\alpha}$:

$$\alpha r r^{\alpha-1} + r^2 \alpha(\alpha-1) r^{\alpha-2} = n^2 r^{\alpha}$$

$$\alpha + \alpha(\alpha-1) = n^2$$

$$\alpha(1 + \alpha - 1) = n^2$$

$$\alpha = \pm n \qquad n \neq 0$$

$G(r) = A r^n + B r^{-n}$ since $|G(0)| < \infty$

we find $B = 0$.

$G(r) = A r^n$

If $n=0$ we have $\frac{d}{dr}\left(r \frac{dG_0}{dr}\right) = 0$

$$r G_0' = C_1$$

$$G_0 = C_1 \ln r + C_2$$

Since $|G(0)|<\infty \Rightarrow C_1=0$ and $G_0(r)=C_2$

Hence we can combine the solutions as

$$r^n \cos(n\theta) \quad \text{for } n=0,1,2$$

$$r^n \sin(n\theta) \quad \text{for } n=1,2,3,\ldots$$

5) <u>Superposition</u>

$$u(r,\theta)=\sum_{n=0}^{\infty} A_n r^n \cos(n\theta)+\sum_{n=1}^{\infty} B_n r^n \sin(n\theta).$$

6) $u(a,\theta)=f(\theta)$

$$=\sum_{n=0}^{\infty} A_n a^n \cos(n\theta)+\sum_{n=1}^{\infty} B_n a^n \sin(n\theta)$$

which is the Fourier-series of $f(\theta)$:

$$A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$n\geq 1$ $$A_n a^n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$B_n a^n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$$