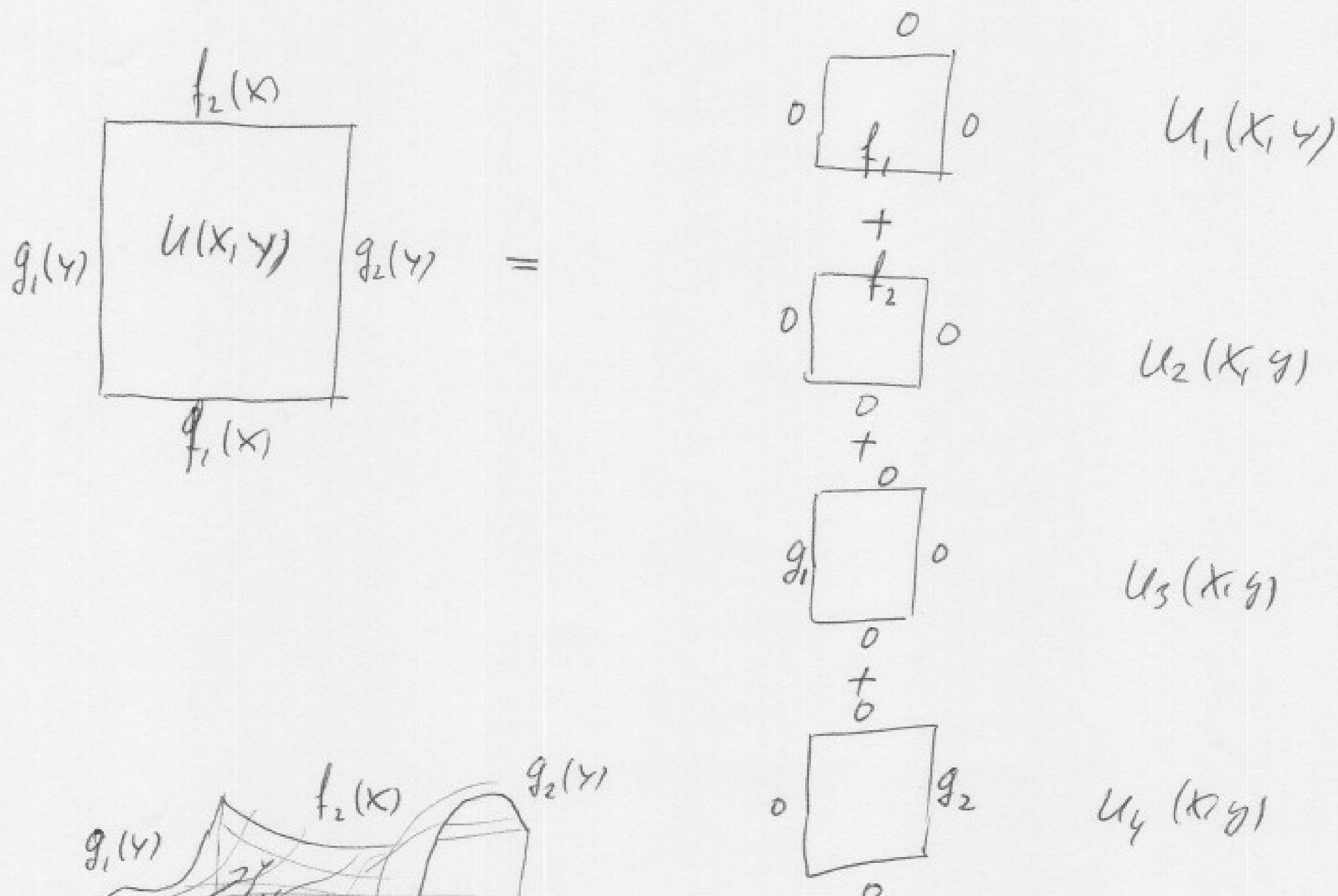
(2.5) Laplace Equation

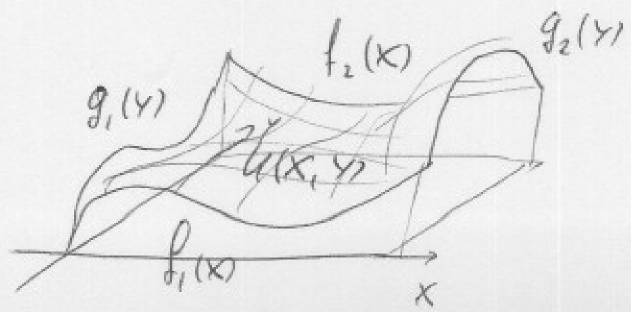
Laplace equation on a rectangle $[0, L] \times [0, H]$.

$\Delta u = 0$ in 2-D:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{on } [0,L]\times[0,H] \\ u(0,y) = g_1(y) \\ u(L,y) = g_2(y) \\ u(x,0) = f_1(x) \\ u(x,H) = f_2(x) \end{cases} \quad \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{boundary conditions}$$

All boundary cond. are non-homogeneous and separation does not work directly.





Split into 4 problems and write

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y),$$

By the principle of superposition, $u(x,y)$ is also a solution to the Laplace equation

$$\Delta u = \Delta u_1 + \Delta u_2 + \Delta u_3 + \Delta u_4 = 0$$

and it satisfies the correct boundary conditions.

At $x=0$: $u(0,y) = u_1(0,y) + u_2(0,y) + u_3(0,y) + u_4(0,y)$

$$= 0 + 0 + g_1(y) + 0$$

$$= g_1(y)$$

etc.

Solve problem 1 (for $u_1(x,y)$)

$$\left.\begin{aligned} &u_{xx} + u_{yy} = 0 \\ &u(0,y) = 0 \\ &u(L,y) = 0 \\ &u(x,0) = f_1(x) \\ &u(x,H) = 0 \end{aligned}\right\}$$

Separation: $u_1(x,y) = h(x)\,\phi(y)$

$$u_{1xx} = h''\cdot\phi, \quad u_{1yy} = h\cdot\phi''$$

$$\Delta u_1 = \phi h'' + h\phi'' = 0 \quad \Rightarrow \quad \boxed{\frac{\phi''}{\phi} = -\frac{h''}{h} = \lambda}$$ separated

boundary conditions:

$u(0,y) = h(0)\,\phi(y) = 0$ if $\phi(y) = 0$ then $u_1 = 0$ trivial sol.

$u(L,y) = h(L)\,\phi(y) = 0$ hence $\phi(y) \neq 0$ and

$$h(0) = 0, \quad h(L) = 0.$$

$u(x,0) = h(x)\,\phi(0) = f_1(x)$

$u(x,H) = h(x)\,\phi(H) = 0$ Since $h(x) \neq 0 \Rightarrow \phi(L) = 0.$

The condition $h(x)\,\phi(0) = f_1(x)$ acts like an initial condition.

<u>X-problem</u>: $\left.\begin{array}{l} h'' = -\lambda h \\ h(0) = 0, \; h(L) = 0 \end{array}\right\}$ proper boundary value problem (2 b. cond.) Sturm-Liouville problem.

<u>Y-problem</u>: $\left.\begin{array}{l} \phi'' = \lambda \phi \\ \phi(H) = 0 \end{array}\right\}$ not proper b.v.p. (1. b. cond.)

<u>Rule</u>: Always solve the most complete boundary value problem first.

<u>X-problem</u>: solutions $h(x) = \sin\left(\frac{n\pi x}{L}\right)$

eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \ldots$

For each of the $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ we need to solve the corresponding Y-problem.

$$\left.\begin{array}{l} \phi'' = \lambda_n \phi_n \\ \phi(H) = 0 \end{array}\right\}$$

Since $\lambda_n > 0$ solutions are given by exponentials.

$$\phi_n(y) = \alpha_n e^{\sqrt{\lambda_n} y} + \beta_n e^{-\sqrt{\lambda_n} y}$$

Boundary cond: $\phi_n(H) = 0 = \alpha_n e^{\sqrt{\lambda_n} H} + \beta_n e^{-\sqrt{\lambda_n} H}$

$$\Longrightarrow \quad \beta_n = -\alpha_n e^{2\sqrt{\lambda_n} H}$$

Then
$$\phi_n(y) = \alpha_n \left(e^{\sqrt{\lambda_n} y} - e^{2\sqrt{\lambda_n} H} e^{-\sqrt{\lambda_n} y} \right)$$
$$= \alpha_n e^{\sqrt{\lambda_n} H} \left(e^{\sqrt{\lambda_n} y} e^{-\sqrt{\lambda_n} H} - e^{\sqrt{\lambda_n} H} e^{-\sqrt{\lambda_n} y} \right)$$
$$= 2\alpha_n e^{\sqrt{\lambda_n} H} \frac{\left(e^{\sqrt{\lambda_n}(y-H)} - e^{-\sqrt{\lambda_n}(y-H)} \right)}{2}$$
$$= a_n \sinh\left(\frac{n\pi}{L}(y-H)\right)$$

By superposition:

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{L}(y-H)\right) \sin\left(\frac{n\pi x}{L}\right)$$

Finally, we use the last remaining condition, the boundary condition $u_1(x,0) = f_1(x)$:

$$f_1(x) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{L}(-H)\right) \sin\left(\frac{n\pi}{L} x\right)$$

Hence $b_n = a_n \sinh\left(-\frac{n\pi H}{L}\right)$ is the Fourier sine series coefficient of f_1:

$$a_n \sinh\left(-\frac{n\pi H}{L}\right) = \frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

or

$$a_n = \frac{2}{L\sinh\left(-\frac{n\pi H}{L}\right)}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

<u>Problem 2</u> :

$$\left.\begin{aligned} &u_{xx}+u_{yy}=0\\ &u(0,y)=0\\ &u(L,y)=0\\ &u(x,0)=0\\ &u(x,H)=f_2(x)\end{aligned}\right\}$$

Transform this problem as $v(x,y) = u(x, H-y)$

Then $\quad v_{xx} = u_{xx}, \quad v_{yy} = u_{yy}$

$$\Rightarrow \quad \left.\begin{aligned} &v_{xx}+v_{yy}=0\\ &v(0,y) = u(0,H-y)=0\\ &v(L,y) = u(L,H-y)=0\\ &v(x,0) = u(x,H) = f_2(x)\\ &v(x,H) = u(x,0)=0\end{aligned}\right\}$$

This is a type 1 problem. Hence the solution is

$$v(x,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi}{L}(y-H)\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L\sinh\left(-\frac{n\pi H}{L}\right)}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

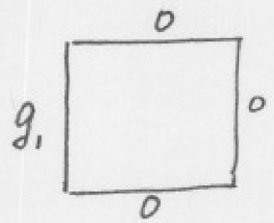
Which gives

$$u_2(x,y) = v(x, H-y)$$

$$= \sum_{n=1}^{\infty} b_n \sinh\left(-\frac{n\pi}{L}y\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L \sinh\left(-\frac{n\pi H}{L}\right)} \int_0^L f_2(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Problem 3: $u_3(x,y)$



Flip x and y and use solution to problem 1:

$$u_3(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{H}(x-L)\right) \sin\left(\frac{n\pi y}{H}\right)$$

$$c_n = \frac{2}{H \sinh\left(-\frac{n\pi L}{H}\right)} \int_0 g_1(y) \sin\left(\frac{n\pi y}{H}\right) dy$$

Problem 4 Flip x and y and use solution of problem 2:

$$u_4(x,y) = \sum_{n=1}^{\infty} d_n \sinh\left(-\frac{n\pi x}{H}\right) \sin\left(\frac{n\pi y}{H}\right)$$

$$d_n = \frac{2}{H \sinh\left(-\frac{n\pi L}{H}\right)} \int_0 g_2(y) \sin\left(\frac{n\pi y}{H}\right) dy$$