Essential Steps for Separation

1) Separation : $u(x,t) = G(t)\,\phi(x)$

2) Substitute into PDE and the homogeneous part of the boundary and initial cond. and derive a

x-problem — t-problem

$$\left[\left.\begin{array}{l} \text{e.g. } \phi'' = -\lambda\phi \\ \quad \phi(0)=0,\ \phi(L)=0 \end{array}\right\} \quad \text{e.g. } \dot{G} = -k\lambda G \right]$$

3) Identify the complete problem of those two and be able to solve it !

$$\left[\text{e.g. } \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \phi_n = \sin\left(\frac{n\pi x}{L}\right)\right]$$

4) Use these λ_n (eigenvalues) and solve the time problem

$$\left[\text{e.g. } G_n(t) = B_n e^{-k\lambda_n t}\right]$$

5) Superposition

$$\left[u(x,t) = \sum_{n=1}^{\infty} a_n \, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) \right]$$

6) Use the nonhomogeneous side condition:

$$\left[\begin{aligned} \text{e.g.} \quad u(x,0) &= f(x) \\ &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \end{aligned} \right]$$

7) Find the Fourier-sine coefficients of f:

$$\left[a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \right]$$