Separation of variables-2 (September 15, Section 2.4)

General scheme for the method of separation of variables for a homogeneous problem

1. Represent the solution as $u(x,t) = \phi(x) G(t)$.
2. Substitute it into the equation, divide it by $\phi(x) G(t)$ (maybe also by some constant), and move all terms with t to one side, and all terms with x to the other.
3. Both sides has to be equal to a constant λ (or $-\lambda$, denote as it is more convenient for you, this will not change the final answer). This splits the problem into two ODEs: for $\phi(x)$ and for $G(t)$. Obtain boundary conditions for $\phi(x)$ from the boundary conditions for u, leave initial conditions for $G(t)$ for later.
4. Solve the eigenvalue problem for $\phi(x)$. This gives a number of eigenvalues λ_n and eigenfunctions $\phi_n(x)$. In general, the eigenfunctions can be multiplied by some constant. This constant usually does not matter, and any nonzero value can be chosen. For example it can be chosen equal to 1.
5. Use in the equation for u special initial conditions $u(x,0) = f(x) = B\phi_n(x)$, then the initial condition for the ODE for $G(t)$ is $G(0) = B$ and $\lambda = \lambda_n$. Solve this ODE for $G_n(t)$.
6. Write down the solution $u(x,t) = \phi_n(x) G_n(t)$ for the special initial conditions $f(x) = B\phi_n(x)$.
7. For general initial condition $f(x) = \sum B_n \phi_n(x)$ we can use the principle of superposition: the solution is the sum of solutions for each term $B_n\phi_n(x)$, that is, $u(x,t) = \sum \phi_n G_n(t)$, $G_n(0) = B_n$.
8. If $f(x)$ is not represented explicitly as a sum $\sum B_n\phi_n(x)$, expand it into the **appropriate** Fourier series.

Neumann boundary conditions

Problem: homogeneous Neuman problem for heat equation (temperature distribution in a thin rod with thermally isolated ends):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} = 0, \quad u(x,0) = f(x).$$

Look for a solution of the form $u(x,t) = \phi(x) G(t)$. Then $\phi(x)\frac{dG(t)}{dt} = kG(t)\frac{d^2\phi(x)}{dx^2}$. Assume $u \neq 0$ and divide by ku:

$$\frac{1}{kG(t)}\frac{dG(t)}{dt} = \frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2} = -\lambda.$$

λ is constant. Boundary conditions: $\phi(0)' G(t) = \phi(L)' G(t) = 0$, so $\phi(0) = \phi(L) = 0$.
Spatial problem:

$$\frac{d^2\phi(x)}{dx^2} = -\lambda\phi(x), \quad 0 < x < L, \quad \phi'(0) = \phi'(L) = 0.$$

We need to find λ for which the problem has a nonzero solution.
Look for a solution in the form $\phi(x) = \exp(\mu x)$, then $\mu^2 = -\lambda$.
1) $\lambda < 0$, $\mu = \pm\sqrt{|\lambda|}$, $\phi(x) = C_1 \exp\left(\sqrt{|\lambda|}x\right) + C_2 \exp\left(-\sqrt{|\lambda|}x\right)$. Boundary conditions:

$$\phi'(x) = C_1\sqrt{|\lambda|}\exp\left(\sqrt{|\lambda|}x\right) - C_2\sqrt{|\lambda|}\exp\left(-\sqrt{|\lambda|}x\right),$$

$$\phi'(0) = C_1\sqrt{|\lambda|} - C_2\sqrt{|\lambda|} = 0, \quad \sqrt{|\lambda|} \neq 0, \quad \Rightarrow \quad C_1 = C_2.$$

$$\phi'(L) = C_1\sqrt{|\lambda|}\exp\left(\sqrt{|\lambda|}L\right) - C_1\sqrt{|\lambda|}\exp\left(-\sqrt{|\lambda|}x\right) = C_1\sqrt{|\lambda|}\left[\exp\left(\sqrt{|\lambda|}L\right) - \exp\left(-\sqrt{|\lambda|}x\right)\right] = 0.$$

$$\exp\left(\sqrt{|\lambda|}L\right) > 1, \quad \exp\left(-\sqrt{|\lambda|}x\right) < 1, \qquad \exp\left(\sqrt{|\lambda|}L\right) - \exp\left(-\sqrt{|\lambda|}x\right) > 0, \quad \Rightarrow \quad C_1 = C_2 = 0.$$

2) $\lambda = 0$, $\frac{d^2\phi(x)}{dx^2} = 0$, $\phi(x) = C_1 x + C_2$. Boundary conditions

$$\phi'(0) = C_1 = 0, \qquad \phi'(L) = C_1 = 0,$$

no restrictions on C_2. Hence we can take as a nonzero solution $\phi_0(x) = 1$.
3) $\lambda > 0$, $\mu = \pm i\sqrt{\lambda}$, $\phi(x) = C_1\sin\left(\sqrt{\lambda}x\right) + C_2\cos\left(\sqrt{\lambda}x\right)$, $\quad \phi'(x) = C_1\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) - C_2\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$.

$$\phi'(0) = C_1\sqrt{\lambda} = 0, \qquad C_1 = 0,$$

$$\phi'(L) = -C_2\sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right) = 0, \qquad \sin\left(\sqrt{\lambda}L\right) = 0, \qquad \sqrt{\lambda}L = \pi n, \qquad n = 1, 2, ...$$

We can combine both cases, $\lambda = 0$ and $\lambda > 0$, by adding $n = 0$ because $\cos(0) = 1 = \phi_0(x)$:

$$\lambda_n = \left(\frac{\pi n}{L}\right)^2, \qquad \phi_n(x) = \cos\left(\frac{\pi n x}{L}\right), \qquad n = \mathbf{0}, 1, 2, 3, \ldots$$

Now let $f(x) = B\phi_n(x) = B\cos\left(\frac{\pi n x}{L}\right)$, then $G(0) = B$. Consider the temporal problem,

$$\frac{dG(t)}{dt} = -k\lambda_n G(t), \quad G(0) = B. \qquad G(t) = B\exp(-k\lambda_n t) = B\exp\left(-k\left(\frac{\pi n}{L}\right)^2 t\right).$$

Conclusion: if the initial temperature profile $f(x) = u(x,0) = B\cos\left(\frac{\pi n x}{L}\right)$, then the solution of the original problem for heat equation is $u(x,t) = B\exp\left(-k\left(\frac{\pi n}{L}\right)^2 t\right)\cos\left(\frac{\pi n x}{L}\right)$.

If initial condition is a sum of **several** cosine terms, for example,

$$u(x,0) = f(x) = A\cos\left(\frac{\pi 3x}{L}\right) + B\sin\left(\frac{\pi 171 x}{L}\right) + C = f_3(x) + f_{171}(x) + f_0(x),$$

then the solution $u(x,t)$ is, according to principle of superposition,

$$u(x,t) = Ae^{-k\lambda_3 t}\cos\left(\frac{\pi 3x}{L}\right) + Be^{-k\lambda_{171} t}\sin\left(\frac{\pi 171 x}{L}\right) + C, \qquad \left(\lambda_0 = 0, \quad e^{-k\lambda_0 t} = 1\right).$$

For $f(x)$ of general kind we represent it as

$$f(x) = B_0 + \sum_{n=1}^{\infty} B_n\cos\left(\frac{\pi n x}{L}\right) = \sum_{n=0}^{\infty} B_n\phi_n(x).$$

Here again Use the fact that $\int_0^L \cos\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi m x}{L}\right)dx = 0$ if $m \neq n$. So we 1) multiply the equality by $\cos\left(\frac{\pi m x}{L}\right)$ and 2) integrate from 0 to L, then only one term with $n = m$ from the infinite sum remains

$$\int_0^L f(x)\cos\left(\frac{\pi m x}{L}\right)dx = \frac{L}{2}B_m, \qquad m = 1, 2, 3..., \qquad \int_0^L f(x)\,dx = LB_0,$$

This gives expression for B_m

$$B_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi m x}{L}\right)dx, \qquad m = 1, 2, 3, ..., \qquad B_0 = \frac{1}{L}\int_0^L f(x)\,dx.$$

The corresponding solution of our problem for heat equation

$$u(x,t) = B_0 + \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{\pi n}{L}\right)^2 t}\cos\left(\frac{\pi n x}{L}\right)$$

Example: solve

$$\frac{\partial u}{\partial t} = \sqrt{5}\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \quad \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(2,t)}{\partial x} = 0, \qquad u(x,0) = f(x) = 15\cos(\pi x) + 16\cos(5\pi x).$$

Here

$$B_2 = 15, \qquad B_{10} = 16, \qquad B_n = 0, \quad n \neq 2,\ 10.$$

$$u(x,t) = B_2 e^{-k\lambda_2 t}\phi_2(x) + B_{10}e^{-k\lambda_{10} t}\phi_2(x) = 15\exp\left(-\sqrt{5}\pi^2 t\right)\cos(\pi x) + 16\exp\left(-25\sqrt{5}\pi^2 t\right)\cos(5\pi x).$$

Problem to think about: (periodic boundary conditions)

$$\frac{d^2\phi(x)}{dx^2} = -\lambda\phi(x), \quad -L < x < L, \quad \phi(-L) = \phi(L), \quad \phi'(-L) = \phi'(L). \qquad \lambda_n = ?, \quad \phi_n(x) = ?$$