(5.7) Nonuniform String

$$\left.\begin{aligned} \rho(x) u_{tt} &= T_0 u_{xx} \quad \text{on } [0,L] \\ u(0,t) &= 0 \\ u(L,t) &= 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\}$$

Separation: $u(x,t) = \phi(x) h(t)$

$$\Rightarrow \qquad h'' = -\lambda h$$

$$\left.\begin{aligned} T_0 \phi'' + \lambda \rho(x) \phi &= 0 \\ \phi(0) &= 0 \\ \phi(L) &= 0 \end{aligned}\right\}$$

The x-problem is a regular SL-problem with $p = T_0$, $q = 0$, $\sigma = \rho$, $\left[-p\phi\phi'\right]_0^L = 0$.

Hence $\lambda_1 \geq 0$.

As before $\lambda_1 = 0$ is impossible (otherwise $\phi_1 = 0$), hence $\boxed{\lambda_1 > 0}$

Call the eigenvalues $\lambda_n > 0$, eigenfunctions ϕ_n.

The time equation $h'' = -\lambda_n h$ has solutions

$$h_n(t) = a_n \cos(\sqrt{\lambda_n}\, t) + b_n \sin(\sqrt{\lambda_n}\, t)$$

And superposition gives

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(\sqrt{\lambda_n}\, t) + b_n \sin(\sqrt{\lambda_n}\, t) \right) \phi_n(x)$$

Initial conditions

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-a_n \sqrt{\lambda_n} \sin(\sqrt{\lambda_n}\, t) + b_n \sqrt{\lambda_n} \cos(\sqrt{\lambda_n}\, t) \right) \phi_n$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

$$\Rightarrow \quad a_n = \frac{\int_0^L f(x)\, \phi_n(x)\, dx}{\int_0^L \phi_n^2(x)\, dx}$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n \sqrt{\lambda_n}\, \phi_n(x)$$

$$\Rightarrow \quad b_n = \frac{1}{\sqrt{\lambda_n}} \frac{\int_0^L g(x)\, \phi_n(x)\, dx}{\int_0^L \phi_n^2(x)\, dx}$$

(5.8) Robin boundary conditions (third kind)

$$\left.\begin{aligned} &u_t = k u_{xx} \\ &u(0,t) = 0 \\ &u_x(L,t) + h\, u(L,t) = 0 \end{aligned}\right\} \quad h>0 \text{ (physical case)}$$

Separation: $u(x,t) = G(t)\,\phi(x)$

$$G'(t) = -\lambda k G$$

$$\left.\begin{aligned} &\phi''(x) + \lambda\phi = 0 \\ &\phi(0) = 0, \quad \phi'(L) + h\phi(L) = 0 \end{aligned}\right\}$$

The x problem is a regular SL-problem with

$p=1,\ q=0,\ \sigma=1$

$$\left[-p\phi\phi'\right]_0^L \underset{\geq 0}{=} -p\phi(L)\phi'(L) = -\phi(L)(-h\phi(L)) = h\phi^2(L)$$

$\Rightarrow \quad \lambda_1 \geq 0.$

$$\lambda = \frac{h\phi^2(L) + \int_0^L (\phi')^2\,dx}{\int_0^L \phi^2\,dx}$$

hence ~~ϕ~~ $\lambda = 0$ is not possible,

$\Rightarrow \quad \lambda_1 > 0.$

Solve the x-problem
$$\left.\begin{aligned} &\phi'' + \lambda\phi = 0 \\ &\phi(0) = 0 \\ &\phi'(L) + h\phi(L) = 0 \end{aligned}\right\}$$

Case 1 $\lambda > 0$: $\phi(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$

$$\phi'(x) = -a\sqrt{\lambda}\sin(\sqrt{\lambda}\,x) + b\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$$

$$\phi(0) = 0 \Rightarrow \quad 0 = a$$

$$\phi'(x) = b\sqrt{\lambda}\cos(\sqrt{\lambda}\,x) \qquad \phi(x) = b\sin(\sqrt{\lambda}\,x)$$

$$\phi'(L) = -h\phi(L)$$

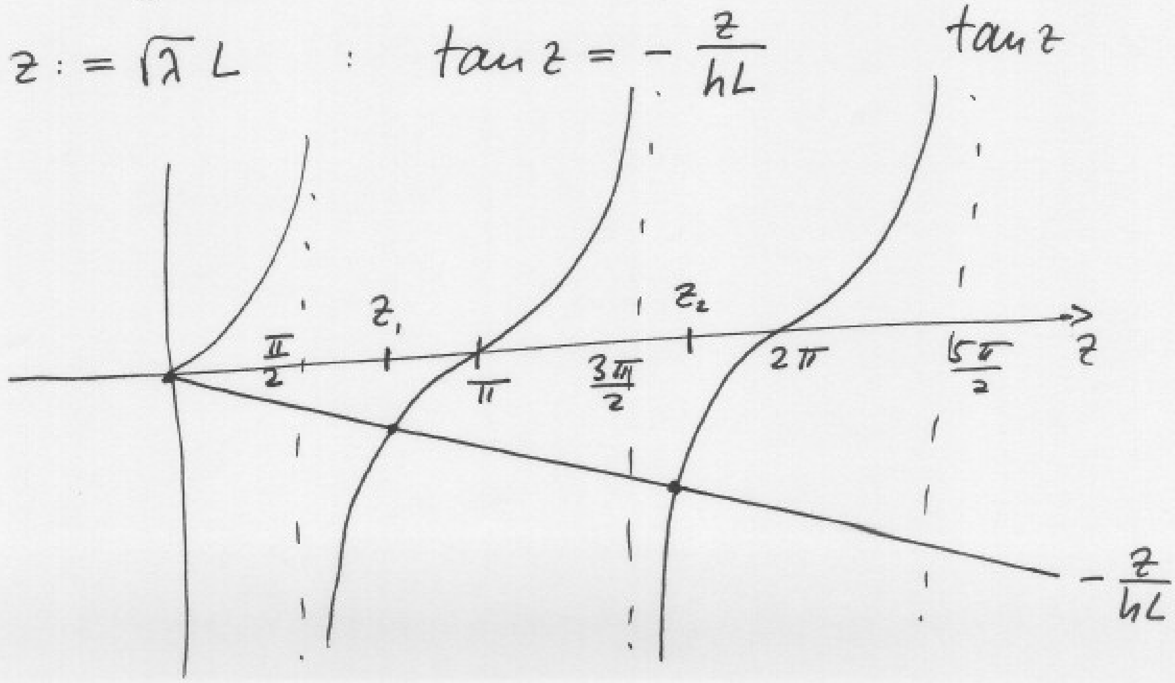
$$b\sqrt{\lambda}\cos(\sqrt{\lambda}L) = -h\,b\sin(\sqrt{\lambda}L)$$

$$\frac{\sin(\sqrt{\lambda}L)}{\cos(\sqrt{\lambda}L} = -\frac{\sqrt{\lambda}}{h}$$

$$\tan(\sqrt{\lambda}L) = -\frac{\sqrt{\lambda}}{h}$$

This is a transcendental equation that cannot be solved directly. We use a graphical method.

$$z := \sqrt{\lambda}L \quad : \quad \tan z = -\frac{z}{hL}$$



From the Rayleigh quotient we know already that $\lambda = 0$ is <u>not</u> an eigenvalue.

We obtain a series of intersections $z_1, z_2, z_3, \dots$ with corresponding eigenvalues

$$\lambda_k = \left(\frac{z_k}{L}\right)^2.$$

For large k the intersections get closer to the corresponding vertical asymptote:

$$z_k \sim \frac{2k-1}{2}\pi \qquad k = 1, 2, 3, \dots$$

$$\Rightarrow \quad \lambda_k \sim \left(\frac{\frac{2k-1}{2}\pi}{L}\right)^2$$

eigenfunction $\quad \phi_k(x) = b \sin(\sqrt{\lambda_k}\, x)$

<u>case 2</u>: $\lambda = 0$ not possible due to Rayleigh quotient.

<u>case 3</u>: $\lambda < 0$:

~~Solve the X problem:~~ $\cancel{\phi'' + \lambda\phi = 0}$

$\cancel{\phi(0) = 0}$

$\cancel{\phi'(L) + h\phi(L) = 0}$

<u>case 1</u> $\lambda < 0$: $\phi(x) = a e^{\sqrt{|\lambda|}x} + b e^{-\sqrt{|\lambda|}x}$

$$\phi(0) = 0 \Rightarrow \quad a = -b$$

$$\Rightarrow \phi(x) = 2a \frac{\left(e^{\sqrt{|\lambda|}x} - e^{-\sqrt{|\lambda|}x}\right)}{2}$$

$$= A \sinh(\sqrt{|\lambda|}x).$$

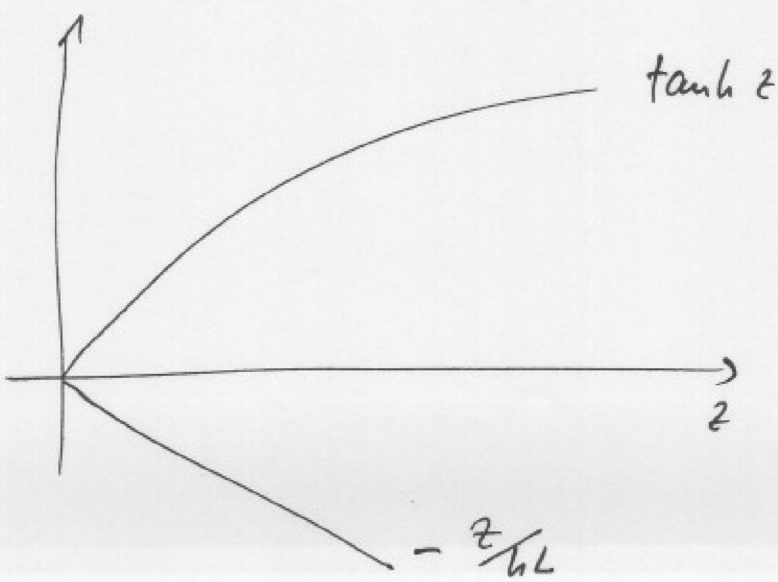
$$\phi' = A\sqrt{|\lambda|} \cosh(\sqrt{|\lambda|}x)$$

$$\phi'(L) = A\sqrt{|\lambda|} \cosh(\sqrt{|\lambda|}L) = -h\,\phi(L)$$

$$= -A \sinh(\sqrt{|\lambda|}L)$$

$$\frac{\sinh(\sqrt{|\lambda|}L)}{\cosh(\sqrt{|\lambda|}L)} = -\frac{\sqrt{|\lambda|}}{h}$$

$$\tanh z = -\frac{z}{hL} \qquad z = \sqrt{|\lambda|}L$$



no intersection for $z>0$.

Summary: $\lambda_k = \left(\frac{z_k}{L}\right)^2, \quad \phi_k = b \sin(\sqrt{\lambda_k}\, x)$

Time equation: $G_n(t) = e^{-\lambda_n k t} \cdot c_1$

Superposition:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n k t} \sin(\sqrt{\lambda_n}\, x)$$

$$\lambda_n = \left(\frac{z_n}{L}\right)^2, \qquad \tan z_n = -\frac{z_n}{hL}.$$