5 Sturm Liouville Eigenvalue Problems

(5.1) Examples

(i) Heat equation with homogeneous Neumann b.c. leads to the spatial eigenvalue problem

$$\left.\begin{array}{l}\phi''(x) = -\lambda\,\phi(x) \\ \phi'(0) = 0,\quad \phi'(L) = 0\end{array}\right\} \begin{array}{l}\text{eigenvalues } \lambda_n = \left(\frac{n\pi}{L}\right)^2 \\ \text{eigenfct. } \quad \phi_n = \cos\left(\frac{n\pi x}{L}\right) \\ n = 0, 1, 2, 3, \ldots\end{array}$$

(ii) Wave equation with homogeneous Dirichlet b.c. leads to

$$\left.\begin{array}{l}\phi''(x) = -\lambda\,\phi(x) \\ \phi(0) = 0,\quad \phi(L) = 0\end{array}\right\} \begin{array}{l}\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3 \\ \phi_n = \sin\left(\frac{n\pi x}{L}\right)\end{array}$$

(iii) Mixed homogeneous b.c.

$$\left.\begin{array}{l}\phi''(x) = -\lambda\,\phi(x) \\ \phi(0) = 0,\quad \phi'(L) = 0\end{array}\right\} \begin{array}{l}\lambda_n = \left(\frac{2n+1}{2}\,\frac{\pi}{L}\right)^2 \quad n = 1, 2, \ldots \\ \phi_n = \sin\left(\frac{2n+1}{2}\,\frac{\pi}{L}\right)\end{array}$$

(iv)

$$\left.\begin{array}{l}\phi''(x) = -\lambda\,\phi(x) \\ \phi'(0) = 0,\quad \phi(L) = 0\end{array}\right\} \begin{array}{l}\lambda_n = \left(\frac{2n+1}{2}\,\frac{\pi}{L}\right)^2 \quad n = 1, 2, \ldots \\ \phi_n = \cos\left(\frac{2n+1}{2}\,\frac{\pi}{L}x\right)\end{array}$$

(5.3) Regular Sturm-Liouville eigenvalue problems

Definition: A regular Sturm-Liouville eigenvalue problem is the problem to find eigenvalues λ and eigenfunctions ϕ for, on $[a,b]$:

$$\left.\begin{aligned} &\frac{d}{dx}\left(p(x)\frac{d\phi(x)}{dx}\right) + q(x)\phi(x) + \lambda\,\sigma(x)\phi(x) = 0 \\ &\beta_1\,\phi(a) + \beta_2\,\frac{d\phi}{dx}(a) = 0 \\ &\beta_3\,\phi(b) + \beta_4\,\frac{d\phi}{dx}(b) = 0 \end{aligned}\right\}$$

$p(x), q(x), \sigma(x)$ are real and continuous, $p(x) > 0$, $\sigma(x) > 0$
$\beta_i \in \mathbb{R}$.

Above examples: $p = 1$, $q = 0$, $\sigma = 1$

(i) $\beta_1 = 0$, $\beta_3 = 0$, $\beta_2 = 1$, $\beta_4 = 1$

(ii) $\beta_1 = 1$, $\beta_3 = 1$, $\beta_2 = 0$, $\beta_4 = 0$

etc.

Mega-Theorem:

1. All eigenvalues λ_n are real.
2. There exists an infinite number of e-values

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots$$

There is a smallest eigenvalue, λ_1, called leading eigenvalue, There is no largest e-value: $\lim_{n\to\infty} \lambda_n = +\infty$.

3. For each λ_n there is a corresponding eigenfunction $\phi_n(x)$ (unique up to multiplication) that has exactly $n-1$ zeroes in $a < x < b$.

4. The set $\{\phi_n(x), n = 1, 2, 3, \ldots\}$ forms an orthogonal basis of PWS[a,b], i.e. each function $f(x) \in PWS[a,b]$ can be written by a generalized Fourier-series

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$

If the generalized F-series converges, then it converges to $f(x)$ where f is continuous, or to $\frac{1}{2}(f(x+) + f(x-))$ at jumps.

5. Eigenfunctions are orthogonal relative to the weight function $\sigma(x)$:

$$\int_a^b \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx = 0 \quad \text{for } \lambda_n \neq \lambda_m$$

6. Eigenvalue λ_n and eigenfunction ϕ_n can be related by the Rayleigh quotient

$$\lambda_n = \frac{-p\phi_n \frac{d\phi_n}{dx}\Big]_a^b + \int_a^b p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2 \, dx}{\int_a^b \phi^2 \sigma \, dx}$$

Example: Heat flow in a non-uniform rod

$$\left.\begin{aligned} c(x)\rho(x)\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + \alpha(x)u \\ u(a,t) = 0, &\quad u(b,t) = 0 \end{aligned}\right\}$$

Separation: $u(x,t) = \phi(x)\,h(t)$

$$c\rho h'\phi = \frac{d}{dx}\left(K_0(x)\frac{d}{dx}\phi(x)\right)h(t) + \alpha\phi h$$

$$\frac{h'}{h} = \frac{1}{c\rho\phi}\frac{d}{dx}\left(K_0\frac{d}{dx}\phi\right) + \frac{\alpha}{c\rho} = -\lambda$$

Spatial problem:

$$\left.\begin{aligned} \frac{d}{dx}\left(K_0\frac{d}{dx}\phi\right) + \alpha\phi + \lambda c\rho\phi = 0 \\ \phi(a) = 0, \quad \phi(b) = 0 \end{aligned}\right\}$$

a regular SL-problem with $p = K_0$, $q = \alpha$

weight function $\sigma(x) = c(x)\rho(x)$.

Generalized F-series

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

a_n: generalized Fourier coefficients.

We find them (again) through orthogonality.

We multiply $f(x)$ by $\phi_m(x)\sigma(x)$ and integrate

$$\int_a^b \phi_m(x) f(x)\, \sigma(x)\, dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_a^b \phi_m(x)\, \phi_n(x)\, \sigma(x)\, dx}_{=0 \text{ for } m \neq n}$$

$$= a_m \int_a^b \phi_m^2(x)\, \sigma(x)\, dx$$

Hence we find:

$$a_n = \frac{\int_a^b \phi_n(x) f(x)\, \sigma(x)\, dx}{\int_a^b \phi_n^2(x)\, \sigma(x)\, dx}$$

Rayleigh quotient

earlier examples: $\phi'' + \lambda\phi = 0$

$p = 1,\ q = 0,\ \sigma = 1.$ In that case

$$\lambda_n = \frac{\int_0^L \left(\frac{d\phi_n}{dx}\right)^2 dx}{\int_0^L \phi_n^2\, dx}$$

Certainly $\lambda_n \geq 0$ (without solving the equation!)

Is also $\lambda > 0$? Assume Dirichlet b. c. $\phi(0) = 0$
$\phi(L) = 0.$

If $\lambda = 0$ is an eigenvalue, then

$$\frac{d\phi_0}{dx} = 0 \quad \text{hence} \quad \phi_0(x) = \text{const.}$$

Since $\phi(0) = 0,\ \phi(L) = 0 \Rightarrow \phi_0(x) = 0.$
Hence ϕ_0 is <u>not</u> an eigenfunction.

This means $\lambda \neq 0$. Hence $\boxed{\lambda > 0}$ (for Dirichlet b.c.)