Example 3 $\quad \dfrac{\partial w}{\partial t} - x\dfrac{\partial w}{\partial x} = 0 \qquad 0 < t < \infty, \quad -\infty < x < \infty$

$$w(x,0) = x^3 - 1$$

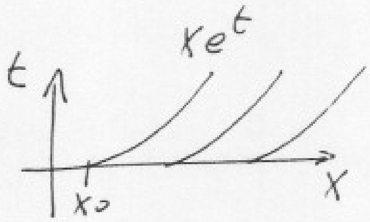
Method of characteristics: $\quad x(t): \quad w = w(x(t), t)$

$$\frac{d}{dt} w(x(t), t) = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial t}$$

need $\dfrac{\partial x}{\partial t} = -x(t) \implies x(t) = x_0 e^{-t} \quad$ or $\quad x_0 = x e^{t}$

Then

$$\frac{d}{dt} w(x(t), t) = 0$$



$$\begin{aligned} w(x(t), t) &= \text{constant} \\ &= w(x(0), 0) \\ &= x_0^3 - 1 \\ &= (xe^t)^3 - 1 = x^3e^{3t} - 1 \end{aligned}$$

solution: $\boxed{w(x,t) = x^3e^{3t} - 1}$

test: $w(x,0) = x^3 - 1$ ✓

$$\frac{\partial w}{\partial t} = 3x^3e^{3t}, \qquad \frac{\partial w}{\partial x} = 3x^2e^{3t}$$

$$\frac{\partial w}{\partial t} - x\frac{\partial w}{\partial x} = 0 \quad ✓$$

Example 4 $\left.\begin{aligned} \frac{\partial z}{\partial t} + 3\frac{\partial z}{\partial x} &= \sin(2\pi t) \\ z(x,0) &= \cos x \end{aligned}\right\}$

Solve this initial value problem on $-\infty < x < \infty$.

$z(x(t), t)$. $\quad \frac{\partial x}{\partial t} = 3 \Rightarrow x(t) = 3t + a$ or $a = x - 3t$.

Then $\frac{d z(x(t),t)}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial t} = \sin(2\pi t)$

$$\Rightarrow \quad z(x(t), t) = -\frac{1}{2\pi}\cos(2\pi t) + C_1$$

$$z(x(0), 0) = -\frac{1}{2\pi} + C_1 = \cos(a)$$

$$= \cos(x - 3t)$$

$$\Rightarrow \quad C_1 = \cos(x-3t) + \frac{1}{2\pi}$$

solution: $\boxed{z(x,t) = -\frac{1}{2\pi}\cos(2\pi t) + \cos(x-3t) + \frac{1}{2\pi}}$

test: $z(x,0) = -\frac{1}{2\pi} + \cos x + \frac{1}{2\pi} = \cos x$ ✓

$$\frac{\partial z}{\partial t} = \sin(2\pi t) + 3\sin(x-3t)$$

$$\frac{\partial z}{\partial x} = -\sin(x-3t)$$

Hence $\frac{\partial z}{\partial t} + 3\frac{\partial z}{\partial x} = \sin(2\pi t)$ ✓

(12.3) Method of Characteristics for the 1-D Wave Equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u(x,t) = 0$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u(x,t) = 0$$

$$\left(\text{or } \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0 \right)$$

now call $w(x,t) := \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u(x,t)$

$$v(x,t) := \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v(x,t)$$

Then we get two first order wave equations

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) w(x,t) = 0$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) v(x,t) = 0$$

We can solve both using the method of characteristics $\quad w(x,t) = P(x+ct)$

$$v(x,t) = Q(x-ct)$$

Hence $u_t + c u_x = P(x+ct)$

$u_t - c u_x = Q(x-ct)$

Solve (i) by the method of characteristics:

Characteristic equations:

$$\frac{\partial x}{\partial t} = c \qquad \frac{du}{dt} = P(x+ct)$$

$$x(t) = ct + a \qquad \frac{du(x(t),t)}{dt} = P(x(t)+ct)$$

$$= P(2ct + a) \cancel{+2t}$$

Let F be an antiderivative of $P(\cancel{2ct+a}\,z)$ w.r.t. z.

$$u(x(t),t) = F(2ct + a) + c_1$$

For (ii) $\quad \frac{\partial x}{\partial t} = -c \qquad \frac{du}{dt} = Q(x-ct)$

$$\tilde{x}(t) = -ct + b \qquad \frac{du}{dt} = Q(-2ct+b)$$

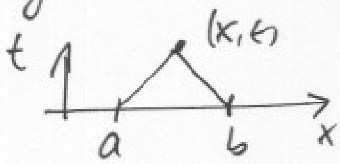
Let $G(z)$ be an antiderivative of $Q(z)$:

$$u(\tilde{x}(t),t) = G(-2ct+b) + c_2$$

together

$$u(x(t),t) + \tilde{u}(\tilde{x}(t),t) = F(2ct+a) + G(-2ct+b) + c_1 + c_2$$

Given (x,t), then $a = x - ct \qquad 2ct + a = x + ct$



$$b = x + ct \qquad -2ct + b = x - ct$$

$$u(x,t) = \frac{1}{2}\Big(F(x+ct) + G(x-ct)\Big) + c_1 + c_2$$

It turns out that we can set $c_1 = c_2 = 0$

and we can include $\frac{1}{2}$ into F and G:

General solution:

$$\boxed{u(x,t) = F(x+ct) + G(x-ct)}$$

Note that D'Alemberts solution can be written in this form using

$$F(z) = \frac{f(z)}{2} + \frac{1}{2c}\int_0^z g(s)\,ds$$

$$G(z) = \frac{f(z)}{2} - \frac{1}{2c}\int_0^z g(s)\,ds$$