(12.2)

First-order PDE's, Method of Characteristics

A linear first-order PDE

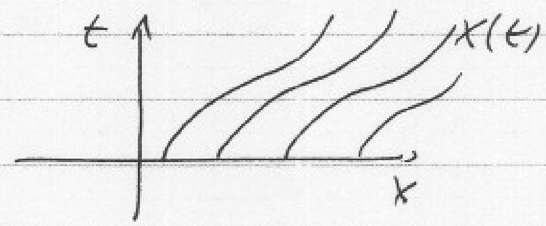
$$\frac{\partial z(x,t)}{\partial t} - c\,\frac{\partial z(x,t)}{\partial x} = 0 \qquad (1)$$

$$z(x,0) = f(x)$$

$$\text{for } -\infty < x < +\infty, \quad t \geq 0$$

no boundary conditions needed, since we work on an unbounded domain.

<u>Idea</u>: Find curves $x(t)$ in the (x,t)-plane, such that the PDE can be reduced to an ODE on these curves.
$x(t)$: <u>characteristics</u>.



Assume we can write a solution like this:

$$z = z(x(t), t).$$

The chain-rule gives $\displaystyle \frac{dz}{dt}(x(t),t) = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial t}$

Compare this to equation (1).
If we choose $x(t)$ to satisfy $\displaystyle \frac{\partial x}{\partial t} = -c$,

then we get $\displaystyle \frac{dz}{dt}(x(t),t) = -c\frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} = 0$

Hence we have transformed one PDE (1) into two ODE's (<u>characteristic equations</u>)

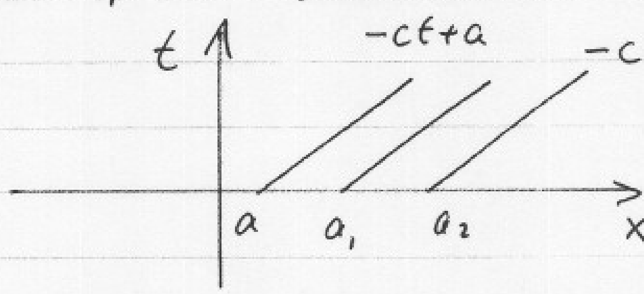
$$\frac{\partial x(t)}{\partial t} = -c \quad , \quad \frac{dz(x(t),t)}{dt} = 0$$

Now we solve the first one:

$$x(t) = -ct + a \qquad a \text{ constant.}$$

a is the initial point of the characteristic curve
$-c$ is it's slope:

$$a = x + ct$$



Now, solve the second characteristic equation:

$$\frac{d}{dt} z(x(t),t) = 0 \Rightarrow z(x(t),t) = \text{const.}$$

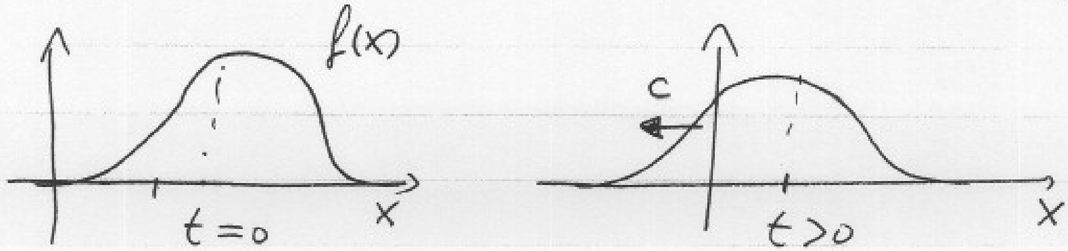
In particular $z(x(t),t) = z(x(0),0) = f(x(0)) = f(a)$

$$= f(x+ct)$$

Now assume (x,t) is given, then

Solution: $\boxed{z(x,t) = f(x+ct).}$

Which is a transition to the left with velocity c:



Example 1: Consider

$$\left.\begin{aligned} &\frac{\partial z}{\partial t} + 5\frac{\partial z}{\partial x} = 0 \\ &z(x,0) = e^{-x^2} \end{aligned}\right\}$$

(i) Write down the characteristic equations and solve them

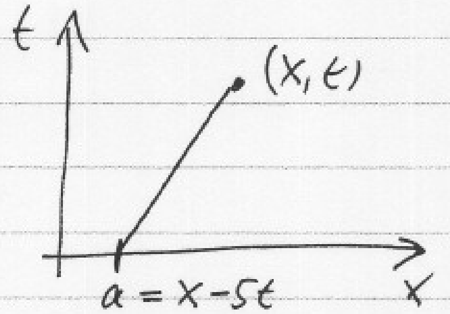
(ii) Given (x,t), find the solution $z(x,t)$.

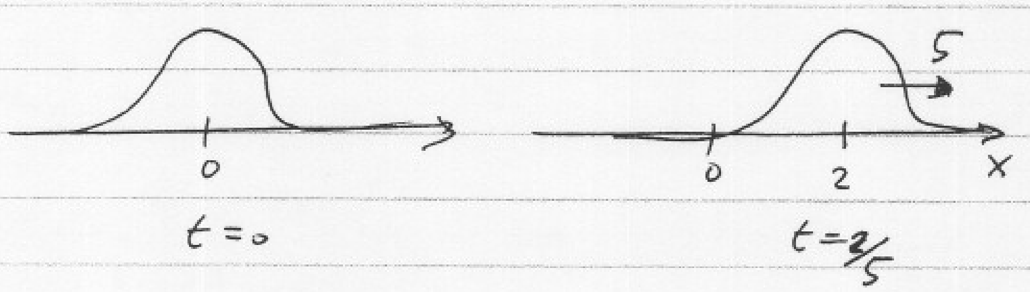
(i) $\dfrac{dx(t)}{dt} = 5 \quad \Rightarrow \quad x(t) = 5t + a$, or $a = x - 5t$

$\dfrac{dz(x(t),t)}{dt} = 0 \quad \Rightarrow \quad z(x(t),t) = f(a) = e^{-(x-5t)^2}$

(ii) Given (x,t) then

$z(x,t) = e^{-(x-5t)^2}$





Now with source (or sink) terms:

$$\boxed{\frac{\partial u(x,t)}{\partial t} + \alpha \frac{\partial u(x,t)}{\partial x} + \beta u(x,t) = 0} \qquad (2)$$

Again, we look for solutions $u(x(t), t)$.
If $u(x(t), t)$ is a solution, then from the chain-rule

$$\frac{d}{dt} u(x(t), t) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial t}$$

it follows that if $\frac{dx}{dt} = \alpha$, then $\frac{d}{dt} u(x(t), t) = -\beta u(x(t), t)$.

Hence we get the two <u>characteristic equations</u>

$$\frac{dx(t)}{dt} = \alpha, \qquad \frac{d}{dt} u = -\beta u \quad , \; u(t) = u(x(t), t)$$

Then $x(t) = \alpha t + a$ $\qquad u(t) = u(0)e^{-\beta t}$

Given (x, t), then $a = x - \alpha t$ and

$$u(x,t) = u(x(0), 0)e^{-\beta t} = \underline{\underline{f(x-\alpha t)e^{-\beta t}}}$$

<u>Example 2:</u> Solve

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + \sqrt{3}\frac{\partial u}{\partial x} - 16u = 0 \\ &u(x,0) = \sin^2(x) \end{aligned}\right\}$$

Just apply the above procedure:

Characteristic equations

$$\frac{dx(t)}{dt} = \sqrt{3} \quad , \quad x(t) = \sqrt{3}\, t + a \quad , \quad a = x - \sqrt{3}\, t$$

$$\frac{du(t)}{dt} = 16u(t) , \quad u(t) = u(0)\, e^{16t}$$

Solution $$u(x,t) = \sin^2(x - \sqrt{3}\, t)\, e^{16t}$$