MATH 300
Advanced Boundary Value Problems I
Formula Sheet

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Table of Integrals

$$\int \sin \omega x \sin \mu x \, dx = \frac{\sin(\mu - \omega)x}{2(\mu - \omega)} - \frac{\sin(\mu + \omega)x}{2(\mu + \omega)} \quad (\omega \neq \mu)$$

$$\int \sin \omega x \cos \mu x \, dx = \frac{\cos(\mu - \omega)x}{2(\mu - \omega)} - \frac{\cos(\mu + \omega)x}{2(\mu + \omega)} \quad (\omega \neq \mu)$$

$$\int \cos \omega x \cos \mu x \, dx = \frac{\sin(\mu - \omega)x}{2(\mu - \omega)} + \frac{\sin(\mu + \omega)x}{2(\mu + \omega)} \quad (\omega \neq \mu)$$

$$\int \sin^2 \omega x \, dx = \frac{x}{2} - \frac{\sin 2\omega x}{4\omega}$$

$$\int \sin \omega x \cos \omega x \, dx = \frac{\sin^2 \omega x}{2\omega}$$

$$\int \cos^2 \omega x \, dx = \frac{x}{2} + \frac{\sin 2\omega x}{4\omega}$$

$$\int x \sin \omega x \, dx = \frac{\sin \omega x}{\omega^2} - \frac{x \cos \omega x}{\omega}$$

$$\int x \cos \omega x \, dx = \frac{\cos \omega x}{\omega^2} + \frac{x \sin \omega x}{\omega}$$

$$\int x^2 \sin \omega x \, dx = \frac{2 \cos \omega x}{\omega^3} + \frac{2x \sin \omega x}{\omega^2} - \frac{x^2 \cos \omega x}{\omega}$$

$$\int x^2 \cos \omega x \, dx = -\frac{2 \sin \omega x}{\omega^3} + \frac{2x \cos \omega x}{\omega^2} + \frac{x^2 \sin \omega x}{\omega}$$

$$\int e^{ax} \sin \omega x \, dx = \frac{e^{ax}(a \sin \omega x - \omega \cos \omega x)}{a^2 + \omega^2}$$

$$\int e^{ax} \cos \omega x \, dx = \frac{e^{ax}(a \cos \omega x + \omega \sin \omega x)}{a^2 + \omega^2}$$

Useful Formulas for Math 300

1. Method of Characteristics for first order equations:

$$u(x,t) = u(x(t),t)$$

   First find $x(t)$, then find $u(x(t),t)$.

2. D'Alemberts formula for the wave equation on $(-\infty, \infty)$:

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds.$$

3. Fourier series on $[-L, L]$:

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L) \\ a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\, dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)\, dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)\, dx \end{aligned}$$

4. Fourier cosine series on $[0, L]$:

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) \\ a_0 &= \frac{1}{L}\int_{0}^{L} f(x)\, dx \\ a_n &= \frac{2}{L}\int_{0}^{L} f(x)\cos(n\pi x/L)\, dx \end{aligned}$$

5. Fourier sine series on $[0, L]$:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin(n\pi x/L) \\ b_n &= \frac{2}{L}\int_{0}^{L} f(x)\sin(n\pi x/L)\, dx \end{aligned}$$

6. Separation of variables:
   (a) Write $u(x,t) = X(x) \cdot T(t)$.
   (b) Solve the Sturm-Liouville problem for $X(x)$.
   (c) Solve the corresponding time problem for $T(t)$.
   (d) Use superposition.
   (e) Use the initial conditions.

7. Laplacian in polar coordinates:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

8. Generalized eigenfunction expansion with a weight function $w(x)$:

$$f(x) = \sum_{i=1}^{\infty} \frac{\displaystyle\int_a^b f(t)\phi_i(t)w(t)\,dt}{\displaystyle\int_a^b \phi_i(t)^2 w(t)\,dt}\,\phi_i(x)$$

9. Fourier-integral formula on $(-\infty, \infty)$:

$$f(x) = \int_0^\infty \left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right)\,d\omega$$
$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx$$
$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx$$

10. Fourier-cosine-integral formula on $[0, \infty)$:

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega$$
$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx$$

11. Fourier-sine-integral formula on $[0, \infty)$:

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\,d\omega$$
$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx$$

12. Fourier transform on $(-\infty, \infty)$:

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$$
$$\mathcal{F}^{-1}(\hat{f})(x) = f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega x}\,d\omega$$

13. Fourier cosine transform on $[0, \infty)$:

$$\mathcal{F}_c(f)(\omega) = \hat{f}_c(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx$$
$$f(x) = \int_0^\infty \hat{f}_c(\omega)\cos\omega x\,d\omega$$

14. Fourier sine transform on $[0, \infty)$:

$$\mathcal{F}_s(f)(\omega) = \hat{f}_s(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx$$
$$f(x) = \int_0^\infty \hat{f}_s(\omega)\sin\omega x\,d\omega$$

15. Gauss kernel

$$g(x,t) = \frac{1}{\sqrt{4\pi k t}}\,e^{-\frac{x^2}{4kt}}.$$

16. Error function

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}\,dx.$$