Review - examples

SOLUTIONS

R-Ex1: Solve
$$\left.\begin{array}{l} u_{tt} = 25 u_{xx} + t^2 \sin(x) \qquad 0 \le x \le 2\pi \\ u(0,t) = 0 = u(2\pi, t) \\ u(x,0) = 3\sin(2x) \\ \frac{\partial u}{\partial t}(x,0) = 7\sin(2x) \end{array}\right\}$$

1) homogeneous problem

$$\left.\begin{array}{l} u_{tt} = 25 u_{xx} \\ u(0,t) = 0 = u(2\pi,t) \end{array}\right\} \Rightarrow$$ corresponding Sturm Liouville problem

$$\left.\begin{array}{l} \varphi'' = -\lambda\varphi \\ \varphi(0) = 0 = \varphi(2\pi) \end{array}\right\} \Rightarrow \begin{array}{l} \lambda_n = \frac{n^2}{4} \\ \varphi_n = \sin\left(\frac{n}{2}x\right) \end{array}$$

2) expand $t^2 \sin(x)$ in terms of $\sum \alpha_n \sin\left(\frac{n}{2}x\right)$.

$\alpha_2 = t^2, \quad \alpha_j = 0 \quad j \ne 2.$

3) try $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\left(\frac{n}{2}x\right)$

$$u_{tt} = \sum_{n=1}^{\infty} a_n''(t)\sin\left(\frac{n}{2}x\right)$$

substitute

$$\sum_{n=1}^{\infty} a_n''(t)\sin\left(\frac{n}{2}x\right) = \sum_{n=1}^{\infty} -25\frac{n^2}{4}a_n(t)\sin(nx) + t^2\sin(x)$$

<u>$n \ne 2$</u>: $a_n'' = -25\frac{n^2}{4}a_n \Rightarrow a_n(t) = A_n\cos\left(\frac{5n}{2}t\right) + B_n\sin\left(\frac{5n}{2}t\right)$

$\underline{n=2}$: $a_2''(t) = -25 a_2(t) + t^2$

homogeneous $a_{2_{hom}}(t) = A_2 \cos(5t) + B_2 \sin(5t)$

particular: $a_{2_{part}}(t) = At^2 + Bt + C$

$$2A = -25At^2 - 25Bt - 25C + t^2$$

$$0 = -25A + 1 \Rightarrow A = \frac{1}{25}$$

$$0 = -25B \Rightarrow B = 0$$

$$2A = -25C \Rightarrow C = -\frac{2}{25}A = \frac{-2}{625}$$

$$\Rightarrow \quad a_2(t) = A_2 \cos(5t) + B_2 \sin(5t) + \frac{1}{25}t^2 - \frac{2}{625}$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\tfrac{5n}{2}t\right) + B_n \sin\left(\tfrac{5n}{2}t\right)\right) \sin\left(\tfrac{n}{2}x\right)$$

$$+ \left(\frac{1}{25}t^2 - \frac{2}{625}\right) \sin(x)$$

4) Initial conditions:

$$3\sin(2x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\tfrac{n}{2}x\right) - \frac{2}{625}\sin(x)$$

$$\Rightarrow A_2 = \frac{2}{625}, \quad A_4 = 3, \quad A_j = 0 \quad j \neq 2, 4.$$

$$7\sin(2x) = \frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} \frac{5n}{2} B_n \sin\left(\tfrac{n}{2}x\right)$$

$$\Rightarrow B_4 = 7 \cdot \frac{2}{20} = \frac{7}{10}, \quad B_j = 0 \quad j \neq 2$$

$$\Rightarrow \quad u(x,t) = \frac{2}{625}\cos(5t)\sin(x) + 3\cos(10t)\sin(2x)$$

$$+ \frac{7}{10}\sin(10t)\sin(2x) + \left(\frac{1}{25}t^2 - \frac{2}{625}\right)\sin(x)$$

<u>Re ex 2:</u> Solve

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1 \\ &\frac{\partial u(0,t)}{\partial x} = d = \frac{\partial u(1,t)}{\partial x} \\ &u(x,0) = \cos(\pi x) \end{aligned}\right\}$$

<u>Split:</u> $u(x,t) = w(x,t) + v(x)$

$$\frac{\partial w}{\partial t} = c\frac{\partial^2 w}{\partial x^2} + c\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial w(0,t)}{\partial x} + \frac{\partial v(0)}{\partial x} = d = \frac{\partial w(1,t)}{\partial x} + \frac{\partial v(1)}{\partial x}$$

$$w(x,0) + v(x) = \cos(\pi x)$$

<u>v-problem</u>

$$\left.\begin{aligned} &\frac{\partial^2 v}{\partial x^2} = 0 \\ &\frac{\partial v(0)}{\partial x} = d \\ &\frac{\partial v(1)}{\partial x} = d \end{aligned}\right\}$$

<u>w-problem</u>

$$\left.\begin{aligned} &\frac{\partial w}{\partial t} = c\frac{\partial^2 w}{\partial x^2} \\ &\frac{\partial w(0,t)}{\partial x} = 0 \\ &\frac{\partial w(1,t)}{\partial x} = 0 \\ &w(x,0) = \cos(\pi x) - v(x) \end{aligned}\right\}$$

v-problem: $v(x) = Ax + B$, $v'(x) = A$

$v'(0) = d = v'(1) \Rightarrow A = d$, B not determined.

$\Rightarrow v(x) = Ax + B$ (Note: you could find B from the integral condition $\int_0^1 B\,dx = \int_0^1 \cos(\pi x)\,dx$, but this is <u>not</u> necessary!)

w-problem

$$\left.\begin{aligned} &\frac{\partial w}{\partial t} = 6\,\frac{\partial^2 w}{\partial x^2} \\ &\frac{\partial w(0,t)}{\partial x} = 0 = \frac{\partial w(1,t)}{\partial x} \\ &w(x,0) = f(x) = \cos(\pi x) - dx - B \end{aligned}\right\}$$

We need cosine-series of $f(x)$. $\cos(\pi x)$ is ok.

$-B$ is ok.

$$dx = \sum_{n=0}^{\infty} \alpha_n \cos(n\pi x)$$

$$\alpha_0 = \frac{1}{1}\int_0^1 dx\,dx = \frac{d}{2}$$

$$\alpha_n = \frac{2}{1}\int_0^1 dx\cos(n\pi x)\,dx$$

$$= \frac{2d}{(n\pi)^2}\left((-1)^n - 1\right)$$

$$\Rightarrow \quad \alpha x = \frac{\alpha}{2} + \sum_{n=1}^{\infty} \frac{2\alpha}{(n\pi)^2}\left((-1)^n - 1\right)\cos(n\pi x)$$

$$w(x,0) = \cos(\pi x) - \frac{\alpha}{2} - \sum_{n=1}^{\infty} \frac{2\alpha}{(n\pi)^2}\left((-1)^n - 1\right)\cos(n\pi x) - B$$

$\Rightarrow$

$$w(x,t) = e^{-c\pi^2 t}\cos(\pi x) - \frac{\alpha}{2} - B$$
$$- \sum_{n=1}^{\infty} \frac{2\alpha}{(n\pi)^2}\left((-1)^n - 1\right) e^{-c(n\pi)^2 t}\cos(n\pi x)$$

$\Rightarrow$

$$u(x,t) = w(x,t) + v(x)$$
$$= -\frac{\alpha}{2} + \alpha x + e^{-c\pi^2 t} - \sum_{n=1}^{\infty} \frac{2\alpha}{(n\pi)^2}\left((-1)^n - 1\right) e^{-c(n\pi)^2 t}\cos(n\pi x)$$

R-ex 3: $c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + \alpha u$

Separation $\quad u(x,t) = G(t)\varphi(x)$

$$c(x)\rho(x)\, G'(t)\varphi(x) = \frac{\partial}{\partial x}\left(K_0(x)\, G(t)\varphi'(x)\right) + \alpha G(t)\varphi(x)$$

$$\frac{G'(t)}{G(t)} = \frac{1}{c(x)\rho(x)\varphi(x)}\frac{\partial}{\partial x}\left(K_0(x)\varphi'(x)\right) + \frac{\alpha\varphi(x)}{c(x)\rho(x)\varphi(x)}$$

$$= -\lambda$$

$\Rightarrow$ spatial Sturm-Liouville problem:

$$\frac{\partial}{\partial x}\left(K_0(x)\varphi'(x)\right)+\alpha\varphi(x)+\lambda\, c(x)\rho(x)\varphi(x)=0$$

with $p(x)=K_0(x)$, $q(x)=\alpha$, $\sigma(x)=c(x)\rho(x)$.

Separation of boundary conditions gives

$$\varphi(0)+\varphi'(0)=0 \qquad \alpha_1=1,\ \alpha_2=1$$

$$\varphi(2)-\varphi'(2)=0 \qquad \alpha_3=1,\ \alpha_4=-1\ .$$

<u>R-ex 4</u>: Estimate the leading eigenvalue of

$$\left.\begin{array}{l}\varphi''-x\varphi+\lambda\varphi=0\\ \varphi'(0)+\varphi(0)=0,\ \ \varphi'(1)=0\end{array}\right\}\ \text{on } [0,1].$$

$$-p\varphi\varphi'\Big|_0^1=\varphi(1)\varphi'(1)-\varphi(0)\varphi'(0)=+\varphi^2(0)\geq 0$$

$p=1$

$q(x)=-x\leq 0 \quad\Rightarrow\quad \lambda_1\geq 0.$

For an upper bound we need an estimator

<u>try #1</u>: $\varphi_I(x)=Ax+B$ that <u>must</u> satisfy the b.c.: $A+B=0$ & $A=0$

$\Rightarrow \varphi_I(x)=0$, <u>does not work!</u>

try # II: $\varphi_{II}(x) = Ax^2 + Bx + C$

b.c.: $B + C = 0$ & $2A + B = 0$

$\Rightarrow$ $2A = -B = C$

choose $C = 1 \Rightarrow B = -1, \; A = \frac{1}{2}$

$\varphi_{II}(x) = \frac{1}{2}x^2 - x + 1$

but I choose $\varphi_{II}(x) = x^2 - 2x + 2$

both are good!

$$\lambda_1 \leq R(\varphi_{II}(x)) = \frac{\int_0^1 (\varphi_{II}')^2 + x\,\varphi_{II}^2\,dx}{\int_0^1 \varphi_{II}^2\,dx}$$

$$\int_0^1 (\varphi_{II}')^2 + x\,\varphi_{II}^2\,dx = \int_0^1 (2x-2)^2 + x\,(x^2 - 2x + 2)^2\,dx$$

$$= \cdots \quad \text{some number} = N_1$$

$$\int_0^1 (\varphi_{II})^2\,dx = \int_0^1 (x^2 - 2x + 2)^2\,dx = N_2$$

Finally: $\lambda_1 \leq R(\varphi_{II}(x)) = \frac{N_1}{N_2}$

<u>Note</u>: In the exam the numbers N_1 and N_2 have to be calculated!

Review ex6: Fourier transform method:

a) $\frac{\partial u}{\partial t}(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2}$ on $-\infty < x < \infty$

$u(x,0) = f(x)$

$\Rightarrow \quad u(x,t) = (f * g)(x) \qquad g(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}}$

b) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, on $-\infty < x < \infty$

$$u(x,0) = \begin{cases} 100, & |x| < 1 \\ 0, & |x| > d \end{cases}$$

$$u(x,t) = (f*g)(x) = \int_{-\infty}^{+\infty} f(y) \frac{e^{-\frac{(x-y)^2}{4t}}}{2\sqrt{\pi t}}\, dy$$

$z := \frac{x-y}{2\sqrt{t}} \qquad dz = \frac{-1}{2\sqrt{t}}\, dy$

$y = 1 \Rightarrow z = \frac{x-1}{2\sqrt{t}} \qquad y = -1 \Rightarrow z = \frac{x+1}{2\sqrt{t}}$

$$u(x,t) = \int_{\frac{x-1}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} 100 \frac{e^{-z^2}}{2\sqrt{\pi t}}\, 2\sqrt{t}\, dz$$

$$= 50\left(\operatorname{erf}\left(\frac{x+1}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x-1}{2\sqrt{t}}\right)\right)$$