University of Alberta
Mathematical and Statistical Sciences
Dr. Thomas Hillen
Dr. A. Yessaei November 29, 2006

Math 300, Fall 2006

Review

The final exam will cover the material of the whole course. As a reminder I summarize the keywords from the material before the midterm.

1. **General:** polar, cylindrical, spherical coordinates; steady states, equilibrium solutions; some linear algebra; PWS-functions; Fourier series; Fourier sine series; Fourier cosine series; linear operators; separation of variables for the heat equation and for the wave equation with all kind of boundary conditions; in 1-D and in 2-D; Laplace equation

2. **Calculus of Fourier series:** Differentiation term-by-term; Integration.

3. **Separation, nonhomogeneous equations:**
   Review example 1: Solve

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= 25\frac{\partial^2 u}{\partial x^2} + t^2\sin(x), \qquad 0 \le x \le 2\pi \\ u(0,t) &= 0 = u(2\pi,t) \\ u(x,0) &= 3\sin(2x) \\ \frac{\partial u(x,0)}{\partial t} &= 7\sin(2x) \end{aligned}$$

4. **Separation; nonhomogeneous boundary conditions:**
   Review example 2: Solve

$$\begin{aligned} \frac{\partial u}{\partial t} &= c\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le 1 \\ \frac{\partial u(0,t)}{\partial x} &= d = \frac{\partial u(2\pi,t)}{\partial x} \\ u(x,0) &= \cos(\pi x) \end{aligned}$$

5. **both: nonhomogeneous equation and nonhomogeneous boundary conditions**

6. **Sturm-Liouville eigenvalues problems** Definition; Spectral Theorem;
Review example 3: The equation for heat flow in a non-uniform rod of length 2 with leaking ends is described by

$$
\begin{aligned}
c(x)\rho(x)\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + \alpha u \\
u(0,t) = -\frac{\partial}{\partial x}u(0,t) & \qquad u(2,t) = \frac{\partial}{\partial x}u(2,t),
\end{aligned}
$$

with physical parameter functions $c(x), \rho(x), K_0(x) > 0$, and $\alpha > 0$. Use separation and show that the spatial problem is a Sturm-Liouville problem (Note: you do not need to solve this equation!)

7. **Rayleigh quotient:** You need to be able to write it down!
Review example 4: Estimate the leading eigenvalue of

$$
\begin{aligned}
\varphi'' - x\varphi + \lambda\varphi &= 0 \\
\varphi'(0) + \varphi(0) = 0 & \qquad \varphi'(1) = 0.
\end{aligned}
$$

8. **Generalized Fourier-series, Bessel functions**

9. **Fourier transform** Fourier integral formula, complex formulations; Fourier-transform, Fourier sine and -cosine transforms.
Review example 5: Find (a) the Fourier transform, (b) the Fourier-sine transform and (c) the Fourier cosine transform of

$$
f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 1 \\ 2 & 1 < x < 2 \\ 0 & x > 2 \end{cases}
$$

10. **Fourier transform methods for PDE:** Gauss kernel, heat kernel, convolution formula for the heat equation, D'Alemberts formula for the wave equation.

Review example 6: Use the error-function to solve

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \qquad \infty < x < \infty \\
u(x,0) &= \begin{cases} 100, & |x| < 1 \\ 0 & |x| > 1 \end{cases}
\end{aligned}
$$

11. **Method of Characteristics:**
Review example 7:
Solve the initial value problem for $-\infty < x < \infty$, $t \geq 0$

$$
\begin{aligned}
\frac{\partial w(x,t)}{\partial t} + 5\frac{\partial w(x,t)}{\partial x} &= e^{3t} \\
w(x,0) &= e^{-x^2}
\end{aligned}
$$

12. **D'Alembert solution of the wave equation:**
Review problem 8:
Solve the wave equation for $-\infty < x < \infty$, $t \geq 0$:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= 25\frac{\partial^2 u}{\partial x^2} \\
u(x,0) &= x^2 \\
\frac{\partial u(x,0)}{\partial t} &= 3.
\end{aligned}$$