<u>**Solutions for Math 300 2005 Final**</u>

**Solution 1.** Assume a separation of variable solution $u(r,\theta) = R(r)\,\Theta(\theta)$. The $\theta$-eigenfunctions must satisfy

$$\Theta'' + \lambda\Theta = 0,\ 0 < \theta < \frac{\pi}{2} \text{ with } \Theta(0) = \Theta\left(\frac{\pi}{2}\right) = 0$$

$$\Longrightarrow \Theta_n(\theta) \propto \sin(2n\theta),\ \lambda = 4n^2,\ n \in \mathbb{Z}^+$$

$$\Longrightarrow r^2 R_n'' + rR_n' - 4n^2 R_n = 0 \Longrightarrow R_n(r) = A_n r^{2n} + B_n r^{-2n},$$

where $A_n$ and $B_n$ are free constants. Hence,

$$u(r,\theta) = \sum_{n=1}^{\infty} \left(A_n r^{2n} + B_n r^{-2n}\right)\sin(2n\theta).$$

Application of the boundary conditions implies that $A_n = B_n = 0$ if $n \neq 1$ or $n \neq 2$, and that

$$A_1 + B_1 = 15 \text{ and } 4A_1 + \frac{B_1}{4} = 0 \Longrightarrow A_1 = -1,\ B_1 = 16,$$

$$A_2 + B_2 = 0 \text{ and } 16A_2 + \frac{B_2}{16} = 255 \Longrightarrow A_2 = 16,\ B_2 = -16.$$

Therefore,

$$u(r,\theta) = \left(\frac{16}{r^2} - r^2\right)\sin(2\theta) + 16\left(r^4 - \frac{1}{r^4}\right)\sin(4\theta).$$

**Solution 2.** The given equation is equivalent to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a\exp\left(i\theta x\right)}{a^2+\theta^2}\,d\theta=\exp\left(-a\left|x\right|\right),\ \ a>0,\ \ -\infty<x<\infty.$$

Thus, we need only show

$$\mathcal{F}\left[\exp\left(-a\left|x\right|\right)\right]=\frac{2a}{a^2+\theta^2}.$$

Proceeding directly,

$$\mathcal{F}\left[\exp\left(-a\left|x\right|\right)\right]=\int_{-\infty}^{\infty}\exp\left(-a\left|x\right|-i\theta x\right)\,dx$$

$$=\int_{-\infty}^{0}\exp\left[\left(a-i\theta\right)x\right]\,dx+\int_{0}^{\infty}\exp\left[-\left(a+i\theta\right)x\right]\,dx$$

$$=\left.\frac{\exp\left[\left(a-i\theta\right)x\right]}{a-i\theta}\right|_{-\infty}^{0}-\left.\frac{\exp\left[\left(a+i\theta\right)x\right]}{a+i\theta}\right|_{0}^{\infty}=\frac{1}{a-i\theta}+\frac{1}{a+i\theta}=\frac{2a}{a^2+\theta^2}.$$

**Solution 3.** Let

$$u(x,t) = \frac{x\sin(2\pi t)}{2\pi} + w(x,t).$$

Substitution into the *IBVP* leads to

$$w_{tt} - w_{xx} = 2\pi x \sin(2\pi t), \; 0 < x < 1, \; t > 0, \tag{1}$$

$$w(0,t) = w(1,t) = w(x,0) = 0, \; w_t(x,0) = -x.$$

The spatial eigenfunctions are proportional to $\sin(n\pi x)$ with $n \in \mathbb{Z}^+$. The eigenfunction expansion for $w(x,t)$ is given by

$$w(x,t) = \sum_{n=1}^{\infty} G_n(t)\sin(n\pi x), \tag{2}$$

and the expansion for the forcing term in (1) is given by

$$2\pi x \sin(2\pi t) = 4\pi \sin(2\pi t) \sum_{n=1}^{\infty} q_n \sin(n\pi x)$$

$$\Longrightarrow q_n = \int_0^1 x\sin(n\pi x)\, dx = \left[\frac{\sin(n\pi x)}{(n\pi)^2} - \frac{x\cos(n\pi x)}{n\pi}\right]_0^1 = -\frac{(-1)^n}{n\pi}. \tag{3}$$

Substitution of (2) and (3) into (1) leads to

$$G_n'' + (n\pi)^2 G_n = -\frac{4(-1)^n \sin(2\pi t)}{n}.$$

If $n \neq 2$, the solution for $G_n(t)$, that satisfies $G_n(0) = 0$, is

$$G_n(t) = A_n \sin(n\pi t) - \frac{4(-1)^n \sin(2\pi t)}{n\pi^2(n^2-4)},$$

where $A_n$ is a free constant. If $n = 2$, resonance occurs and the solution for $G_2(t)$, that satisfies $G_2(0) = 0$, is

$$G_2(t) = A_2 \sin(2\pi t) + \frac{t\cos(2\pi t)}{2\pi},$$

where $A_2$ is a free constant. The initial condition $w_t(x,0) = -x$, implies

$$-x = \sum_{n=1}^{\infty} G_n'(0)\sin(n\pi x) \Longrightarrow G_n'(0) = -2\int_0^1 x\sin(n\pi x)\, dx = \frac{2(-1)^n}{n\pi}.$$

And since,

$$G_n'(0) = \begin{cases} n\pi A_n - \frac{8(-1)^n}{n\pi(n^2-4)}, \text{ if } n \neq 2, \\ 2\pi A_2 + \frac{1}{2\pi}, \text{ if } n = 2, \end{cases}$$

it follows that

$$A_n = \begin{cases} \frac{2(-1)^n}{\pi^2(n^2-4)}, \text{ if } n \neq 2, \\ \frac{1}{4\pi^2}, \text{ if } n = 2. \end{cases}$$

Therefore,

$$u(x,t) = \frac{x \sin(2\pi t)}{2\pi} + \left[ \frac{\sin(2\pi t)}{4\pi^2} + \frac{t \cos(2\pi t)}{2\pi} \right] \sin(2\pi x)$$

$$+ \sum_{\substack{n=1 \\ n \neq 2}}^{\infty} \frac{2(-1)^n}{\pi^2 n (n^2-4)} \left[ n \sin(n\pi t) - 2 \sin(2\pi t) \right].$$

**Problem 4.** Let

$$U(\theta,t)=\mathcal{F}[u(x,t)]=\int_{-\infty}^{\infty}u(x,t)\exp(-i\theta x)\ dx,$$

$$F(\theta)=\mathcal{F}[f(x)]=\int_{-\infty}^{\infty}f(x)\exp(-i\theta x)\ dx.$$

The Fourier Transformed pde is

$$\frac{d^2U}{dt^2}+(c\theta)^2U=0,\ \ t>0,\ \ U(\theta,0)=F(\theta)\,,\ \ U_t(\theta,0)=0,$$

the solution of which is

$$U(\theta,t)=F(\theta)\cos(c\theta t)\,,$$

thus

$$\begin{aligned}u(x,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\theta)\cos(c\theta t)\exp(i\theta x)\ d\theta\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\theta)\left(\frac{e^{ic\theta t}+e^{-ic\theta t}}{2}\right)\exp(i\theta x)\ d\theta\\&=\frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\theta)\exp[i\theta(x+ct)]\ d\theta+\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\theta)\exp[i\theta(x-ct)]\ d\theta\right]\\&=\frac{f(x+ct)+f(x-ct)}{2}.\end{aligned}$$