Midterm Solution

Problem 3 $\nabla^2 u = 0 \quad 0 < \theta < \pi, \quad 0 \le r \le 1$

$$\frac{\partial u}{\partial \theta}(r,0) = 0 = \frac{\partial u}{\partial \theta}(r,\pi)$$

$u(1,\theta) =$ $2 + \cos 2\theta$ purple

or $1 + \cos 3\theta$ blue

$|u(0,\theta)| < \infty$

Separation: $r^2 R'' + rR' - \lambda R = 0, \quad |R(0)| < \infty$

$u(r,\theta) = R(r)\,\theta(\theta)$

$$\theta'' = -\lambda\theta$$
$$\theta'(0) = 0 = \theta'(\pi)$$
$$\Rightarrow \lambda_n = n^2, \quad \theta_n(\theta) = \cos(n\theta) \qquad n = 0, 1, 2, \ldots$$

Solve the R-equation as in class with use of

$|u(0,\theta)| < \infty \Rightarrow R_0(r) = 1$

$$R_n(r) = A_n r^n$$

$$\Rightarrow u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta)$$

$$u(1,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta)$$

purple: $A_0 = 2, \; A_2 = 1, \; A_j = 0$ for $j \neq 0, 2$

$$u(r,\theta) = 2 + r^2 \cos 2\theta$$

blue: $A_0 = 1, \; A_3 = 1, \; A_j = 0$ for $j \neq 0, 3$

$$u(r,\theta) = 1 + r^3 \cos 3\theta$$