MATH 300 Fall 2004
Advanced Boundary Value Problems I
Midterm Examination
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A thin, homogeneous bar of length π has its sides poorly insulated, so that heat radiates freely from the bar along its length. Assuming that the heat transfer coefficient A is constant, and that the temperature T of the surrounding medium is also constant, the temperature $u(x,t)$ in the bar at position x and time t satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - A(u - T), \quad 0 < x < \pi, \quad t > 0.$$

The ends of the bar are kept at temperature T, and the initial temperature is $f(x) = x + T$, for $0 < x < \pi$.

(1) State the initial boundary value problem satisfied by $u(x,t)$.

(2) Transform this problem into a familiar problem by setting $v(x,t) = e^{At}[u(x,t) - T]$, and then finding the initial boundary value problem satisfied by $v(x,t)$.

(3) Use the method of separation of variables to solve the problem in part (2), showing **all** of the necessary steps.

SOLUTION:

(1) The initial boundary value problem satisfied by $u(x,t)$ is

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - A(u - T), \quad 0 < x < \pi, \quad t > 0 \\
u(0,t) &= T, \quad t > 0 \\
u(\pi,t) &= T, \quad t > 0 \\
u(x,0) &= x + T, \quad 0 < x < \pi.
\end{aligned}$$

(2) Letting $v = e^{At}(u - T)$, we have

$$\begin{aligned}
\frac{\partial v}{\partial t} &= Ae^{At}(u - T) + e^{At}\frac{\partial u}{\partial t} \\
\frac{\partial^2 v}{\partial x^2} &= e^{At}\frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

so that

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = Ae^{At}(u - T) + e^{At}\frac{\partial u}{\partial t} - e^{At}\frac{\partial^2 u}{\partial x^2} = e^{At}\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + A(u - T)\right) = 0$$

since u is a solution to the original partial differential equation.

Therefore, $v(x,t)$ satisfies the initial boundary value problem

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0$$
$$v(0,t) = 0, \quad t > 0$$
$$v(\pi,t) = 0, \quad t > 0$$
$$v(x,0) = x, \quad 0 < x < \pi.$$

(3) Assuming a solution of the form $v(x,t) = X(x) \cdot T(t)$, we have

$$X \cdot T' = X'' \cdot T,$$

and separating the variables, we get

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda,$$

so that

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < \pi \qquad \text{and} \qquad T'(t) + \lambda T(t) = 0, \quad t > 0.$$

We can satisfy the boundary conditions by requiring that $X(0) = X(\pi) = 0$, so that $X(x)$ satisfies the boundary value problem

$$X'' + \lambda X = 0, \quad 0 < x < \pi$$
$$X(0) = 0$$
$$X(\pi) = 0.$$

If $\lambda = 0$, the general solution to this equation is

$$X(x) = ax + b,$$

and applying the boundary conditions, we find $a = b = 0$, and $X(x) = 0$ for $0 \le x \le \pi$.

Similarly, if $\lambda < 0$, say $\lambda = -\mu^2$ where $\mu \ne 0$, the general solution to the equation is

$$X(x) = a \cosh \mu x + b \sinh \mu x,$$

and applying the boundary conditions, we find $a = b = 0$, and again the solutuion is $X(x) = 0$ for $0 \le x \le \pi$.

If $\lambda > 0$, say $\lambda = \mu^2$ where $\mu \ne 0$, the general solution to the equation in this case is

$$X(x) = a \cos \mu x + b \sin \mu x,$$

and applying the boundary condition $X(0) = 0$ we get $a = 0$. From the second boundary condition, in order to get a nontrivial solution, we need $\sin \mu\pi = 0$, so that μ must be a multiple of π.

The eigenvalues are $\mu_n^2 = n^2$, for $n = 1, 2, \ldots$, and the corresponding eigenfunctions are $X_n(x) = \sin nx$, for $n = 1, 2, \ldots$.

For each $n \ge 1$, the corresponding solution to $T' + n^2 T = 0$ is $T_n(t) = e^{-n^2 t}$, and the normal modes are given by

$$v_n(x,t) = X_n(x) \cdot T_n(t) = e^{-n^2 t} \sin nx$$

for $0 < x < \pi$, and $t > 0$.

Since both the partial differential equation and the boundary conditions are linear and homogeneous, then we can use the superposition priciple to write

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$$

for $0 < x < \pi,\ t > 0$, and $v(x,t)$ satisfies the heat equation and the boundary conditions.

In order to satisfy the initial condition, we set $t = 0$ in this expression to get

$$x = v(x,0) = \sum_{n=1}^{\infty} b_n \sin nx,$$

so that b_n's are the Fourier sine coefficients of the function $v(x,0) = x,\ 0 < x < \pi$, and for $n \geq 1$,

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx = \frac{2}{\pi} \left[-\frac{x \cos nx}{n} \bigg|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right] = \frac{2}{n} (-1)^{n+1}.$$

Therefore,

$$v(x,t) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} e^{-n^2 t} \sin nx$$

for $0 < x < \pi,\ t > 0$, and the temperature in the poorly insulated bar is given by

$$u(x,t) = T + e^{-At} v(x,t) = T + \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} e^{-(n^2 + A)t} \sin nx$$

for $0 < x < \pi,\ t > 0$.