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ON A THEOREM OF ERDŐS AND SZEKERES

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The theorem of Erdos and Szekeres [1] referred to in the title may be stated as follows.

THEOREM A. *Given a sequence of* r *distinct real numbers such that the number of terms of every decreasing subsequence is at most* m, *then there exists an increasing subsequence of more than* n *terms, where* n *is the largest integer less than* r/m.

An extremely simple and elegant proof of the theorem was given by A. Seidenberg [2]. This note is intended to point out that a result analogous to the above holds under a more general setting.

Let S be a finite sequence of elements, $S = \{a_1, a_2, \ldots, a_r\}$ on which a binary relation R is defined such that, for every distinct pair a, b of elements of S, exactly one of the relations $a\,R\,b$, $b\,R\,a$ holds. We do not require R to be transitive - that is, $a\,R\,b$ and $b\,R\,c$ may hold, but not $a\,R\,c$.

A subsequence $\{b_1, b_2, \ldots, b_s\}$ of S is said to be an ascending R - subsequence with first term b_1 provided either $s = 1$, or $s > 1$ and $b_i\,R\,b_{i+1}$ holds for $i = 1, 2, \ldots, s-1$. It is said to be a descending R - subsequence with last term b_1 if either $s = 1$, or if $s > 1$ and $b_i\,R\,b_{i-1}$ holds for $i = s, s-1, \ldots, 2$. Thus only single term subsequences are both ascending and descending R - subsequences. Finally, the "length" of a subsequence is defined to be the number of its elements.

We now have

THEOREM B. *Let* S *be a finite sequence of* r *abstract elements, on which a binary relation* R *is defined such that for distinct elements* a, b *in* S, *exactly one of the properties* $a\,R\,b$, $b\,R\,a$ *holds. If every ascending* R - *subsequence of* S *has a length not exceeding* m, *then there exists a descending* R - *subsequence of length not less than* r/m.

The proof, which is essentially similar to Seidenberg's proof of Theorem A, is as follows:

To each element a_i of S we associate an ordered pair of positive integers (m_i, n_i) where $m_i(n_i)$ is the largest of the lengths of all ascending R - subsequences with first term a_i (descending R - subsequences with last term a_i). These integers m_i, n_i exist for all

i $(i = 1, 2, \ldots, r)$ and are ≥ 1. Further, for any two distinct elements a_i, a_j of S, the corresponding ordered pairs (m_i, n_i) and (m_j, n_j) are distinct, for if $i < j$ and if $a_i R a_j$ holds, we have $m_i \geq m_j + 1$, while, if $a_j R a_i$ holds, we have $n_i \geq n_j + 1$. The total number of ordered pairs is then r, the number of elements in S. Let $t = \operatorname{Max}_{1 \leq i \leq r} n_i$. Since $\operatorname{Max}_{1 \leq i \leq r} m_i \leq m$, it follows that the number of all the possible ordered pairs (m_i, n_i) is $\leq mt$, and hence $r \leq mt$. Thus, $t \geq r/m$ and the theorem is proved.

REFERENCES

1. P. Erdös and G. Szekeres, A combinatorial problem in geometry. Composite Math. 2 (1935) 463-470.

2. A. Seidenberg, A simple proof of a theorem of Erdös and Szekeres. J. London Math. Soc. 34 (1959) 352.

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