Annales Univ. Sci. Budapest., Sect. Comp. 26 (2006) 43-49

A REMARK ON THE PRODUCT PARTITION OF INTEGERS INTO k PARTS

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Abstract. Let $f_k(n)$ be the number of solutions of the equation $n = m_1 m_2 \dots m_k$ in integers $(2 =) m_1 < m_2 < \dots < m_k$. The authors analyze the question how $\sum_{n \le x} f_k(n)$ can be estimated with good remainder terms by using known result.

1. Let $k \in \mathbb{Z}$ be an integer, $f_k(n)$ be the number of solutions of the equation

$$n = m_1 m_2 \dots m_k,$$

in integers $(2 \le) m_1 < m_2 < \dots < m_k$.

Let

$$F(t,s) = \prod_{n=2}^{\infty} \left(1 + \frac{t}{n^s}\right).$$

The research of second author supported in part by a grant from NSERC. He died February 5, 2006.

The research of first author supported by the Applied Number Theory Research Group of the Hungarian Academy of Sciences, the Hungarian National Foundation for Scientific Research under grant OTKA T46993 and in part by a grant from NSERC.

Mathematics Subject Classification: 11M06, 11P99, 11N37

Then

$$\begin{aligned}\log F(t,s) &= \sum_{k=1}^{\infty} \frac{t^k}{k} \cdot (-1)^{k-1} \left(\sum_{n=2}^{\infty} \frac{1}{n^{ks}} \right) = \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} t^k}{k} (\zeta(ks) - 1) = \\ &= t\zeta(s) - t + a(t,s), \\ a(t,s) &= \sum_{k=2}^{\infty} \frac{(-1)^{k-1} t^k}{k} (\zeta(ks) - 1).\end{aligned}$$

It is clear that for $|t| < \sqrt{2}$, the function $a(t,s)$ as a function of s is regular and bounded in Re $s > \frac{1}{2} + \delta$ where δ is an arbitrary positive constant.

Let

$$F_k(s) = \sum_{n=1}^{\infty} \frac{f_k(n)}{n^s}.$$

We have $e^{-t+a(t,s)} = b_0(s) + b_1(s)t + b_2(s)t^2 + \dots,\ b_0(s) = 1$, where $b_\nu(s)$ are bounded in $\sigma > \frac{1}{2} + \delta$, they are polynomials of $\zeta(2s), \dots, \zeta(\nu s)$ for every ν. The explicit form of them can be computed.

Let $x_1 = \log x,\ x_2 = \log x_1, \dots$.

Since

$$F(t,s) = 1 + \sum_{k=1}^{\infty} t^k F_k(s),$$

$$e^{t\zeta(s)} = \sum_{k=0}^{\infty} \frac{\zeta(s)^k}{k!} t^k,$$

therefore

$$F_k(s) = \sum_{\nu=0}^{k} \frac{b_\nu(s)}{(k-\nu)!} \zeta^{k-\nu}(s). \tag{1.1}$$

We would like to estimate

$$S_k(x) = \sum_{n \le x} f_k(n).$$

Since

$$e^{-t+a(t,s)} = \left(\sum \frac{(-t)^{\nu_0}}{\nu_0!}\right) \prod_{k=2}^{\infty} \left(\sum_{m=0}^{\infty} \frac{(-1)^{(k-1)m} t^{km}}{k^m} (\zeta(ks)-1)^m\right),$$

therefore

$$\frac{b_\nu(s)}{(k-\nu)!} = E_1 \cdot G_1(s) + \ldots + E_p \cdot G_p(s), \tag{1.2}$$

where the general form of $G_l(s)$ can be written as

$$G(s) = \zeta^{m_1}(a_1 s) \ldots \zeta^{m_q}(a_q s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \tag{1.3}$$

where $(2 \leq) a_1 < \ldots < a_q$, $m_1, \ldots, m_s$ are positive integers, and

$$m_1 a_1 + \ldots + m_q a_q \leq \nu.$$

Let

$$\begin{aligned} D(s) &= D(s|G) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \\ B(y) &= (B(y|G) =) \sum_{n \leq y} g(n). \end{aligned} \tag{1.4}$$

From (1.3) we have

$$B(y) = \sum_{n_1^{a_1} \ldots n_q^{a_q} \leq x} d_{m_1}(n_1) \ldots d_{m_q}(n_q). \tag{1.5}$$

Lemma 1. *We have*

$$B(y) \leq C x^{1/a_1} x_1^{m_1 - 1} \zeta\left(\frac{a_2}{a_1}\right) \ldots \zeta\left(\frac{a_q}{a_1}\right) \tag{1.6}$$

for $q \geq 2$, *and*

$$B(y) \leq C x^{1/a_1} x_1^{m_1 - 1} \quad \textit{if} \quad q = 1. \tag{1.7}$$

Proof. Since

$$\sum_{n^a\le x} d_m(n) = \sum_{u_1\dots u_m\le x^{1/a}} 1 = \sum_{u_1\dots u_{m-1}\le x^{1/a}} \frac{x^{1/a}}{u_1\dots u_{m-1}} \le$$
$$\le x^{1/a}\left(\sum_{u<x^{1/a}}\frac{1}{u}\right)^{m-1} \le$$
$$\le x^{1/a}\left(\frac{1}{a}x_1 + c\right)^{m-1},$$

therefore the assertion is true for $q = 1$.

Let $q \ge 2$. Assume that the assertion is true for $q-1$.

Then

$$\sum_{n_1^{a_1}\dots n_q^{a_q}\le x} d_{m_1}(n_1)\dots d_{m_q}(n_q) = \sum_{n_2^{a_2}\dots n_q^{a_q}\le x} d_{m_2}(n_2)\dots d_{m_q}(n_q)\sum\nolimits_{n_2,\dots n_q},$$

$$\sum\nolimits_{n_2,\dots,n_q} = \sum_{n_1\le\left(\frac{x}{n_2^{a_2}\dots n_q^{a_q}}\right)^{1/a_1}} d_{m_1}(n_1) \le cx^{1/a_1}x_1^{m_1-1}\cdot\frac{1}{n_2^{a_2/a_1}\dots n_q^{a_q/a_1}},$$

and so

$$B(y) \le cx^{1/a_1}x_1^{m_1-1}\zeta\left(\frac{a_2}{a_1}\right)^{m_2-1}\dots\zeta\left(\frac{a_q}{a_1}\right)^{m_q-1}. \tag{1.8}$$

Lemma 2. *Let*

$$D_k(x) := \sum_{n\le x} d_k(n) = xQ_{k-1}(\log x) + \Delta_k(x),$$
$$Q_{k-1}(\log x) := \operatorname*{Res}_{s=1} x^{s-1}\zeta^k(s)s^{-1}.$$

Let $\alpha_2 \le \alpha_3 \le \dots$ be such a sequence for which $\alpha_2 > \frac{1}{2}$, and for each $\varepsilon > 0$,

$$\Delta_k(x) = O\left(x^{\alpha_k+\varepsilon}\right)$$

holds. Possible value of α_k can be found in [1], Theorem 13.2.

2. Let us estimate

$$E(x \mid G, k, \nu) = \sum_{mn \le x} g(m) d_{k-\nu}(n), \tag{2.1}$$

where G is a function of form (1.3).

Then

$$E(x \mid G, k, \nu) = \sum_{m \le x} g(m) D_{k-\nu}(x/m) =$$

$$= \sum_{m \le x} g(m) \left\{ \frac{x}{m} Q_{k-\nu-1}\left(\log \frac{x}{m}\right) + O\left(\left(\frac{x}{m}\right)^{\alpha_k+\varepsilon}\right)\right\} =$$

$$= \sum_{m \le x} \frac{g(m)}{m} Q_{k-\nu-1}\left(\log \frac{x}{m}\right) + O\left(x^{\alpha_k+\varepsilon} \sum \frac{g(m)}{m^{\alpha_k+\varepsilon}}\right).$$

Since $\alpha_{k-\nu} > 1/2$, therefore

$$\sum \frac{g(m)}{m^{\alpha_{k-\nu}+\varepsilon}} < \infty.$$

Let

$$Q_{k-\nu-1}(y) = \sum_{\mu=0}^{k-\nu-1} e_\mu y^\mu.$$

Therefore

$$\sum_{m \le x} \frac{g(m)}{m} \sum_{\mu=0}^{k-\nu-1} e_\mu (x_1 - (\log m))^\mu = \sum_{h=0}^{k-\nu-1} x_1^h U_h(x),$$

$$U_h(x) = \sum_{h=0}^{k-\nu-l-1} d_{h,l} \sum_{m \le x} \frac{g(m)}{m} (\log m)^l.$$

Let

$$\eta_l := \sum \frac{g(m)}{m} (\log m)^l. \tag{2.2}$$

Since

$$\sum_{m \ge x} g(m) \frac{(\log m)^l}{m} \ll \sum_{t=0}^{\infty} \frac{[\log(2^t x)]^l}{2^t x} \sum_{2^t x < m < 2^{t+1} x} g(m) \ll$$

$$\ll \sum_{t=0}^{\infty} \frac{(\log 2^t x)^l}{\sqrt{2^t x}} (\log x)^{m_1 - 1} \zeta(2)^\nu \ll \frac{(\log x)^K}{\sqrt{x}}$$

holds with a suitable large K, therefore

$$U_h(x) = \sum_{h=0}^{k-\nu-l-1} d_{h,l}\eta_l + O\left(x^{-1/2+\varepsilon}\right),$$

$\varepsilon > 0$ is an arbitrary small positive integer. Let us observe furthermore that in (1.1) $b_0(s) = 1$.

We proved the following

Lemma 3. *For* (2.1) *we have*

$$E(x \mid G, k, \nu) = x\tilde{Q}_{k-\nu-1}(\log x) + O\left(x^{\alpha_k+\varepsilon}\right).$$

From Lemma 3 the following assertion follows.

Theorem 1. *Let $k \geq 2$ be an arbitrary integer. Then*

$$S_k(x) = x\tilde{P}_{k-1}(\log x) + O\left(x^{\alpha_k+\varepsilon}\right),$$

where $\tilde{P}_{k-1}(y) = \pi_{k-1}\, y^{k-1} + \ldots + \pi_0$ is a polynomial, the leading coefficient π_{k-1} satisfies $\pi_{k-1} = \frac{1}{k!}$.

Remarks.

1. A.F. Lavrik [2] counted the coefficients of the polynomials Q_{k-1} in Lemma 2. By his method and by counting the coefficients of the expansions $b_\nu(s) = b_\nu^{(0)} + b_\nu^{(1)}(s-1) + \ldots$, one can determine the coefficients of $\tilde{P}_{k-1}$ in Theorem 1.

2. A.A. Karacuba [3] proved, by using the method of I.M. Vinogradov, that

$$\sum_{n \leq x} d_k(n) = xP_{k-1}(x_1) + O\left(x^{1-\frac{c}{k^{2/3}}+\varepsilon}\right)$$

uniformly as $k/x_2 \leq \varepsilon_x$, where $\varepsilon_x \to 0$ arbitrarily, $\varepsilon > 0$, $c > 0$, the constant implied by the error term is absolute, P_{k-1} is a polynomial of degree $k-1$, the leading term of which is 1.

By using his theorem we can deduce that Theorem 1 remains valid uniformly as $\frac{k}{x_2} \leq \varepsilon_x$, with $\alpha_k = 1 - \frac{c}{k^{2/3}}$.

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(Received November 30, 2004)

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