# IDENTITIES FOR MULTIPLICATIVE FUNCTIONS

M.V. Subbarao[1] and A.A. Gioia[2]

(received November 11, 1965)

1. Introduction. Throughout this paper the arithmetic functions $L(n)$ and $w(n)$ denote respectively the number and product of the distinct prime divisors of the integer $n > 1$, with $L(1) = 0$ and $w(1) = 1$. Also let

$$C(m, n) = \begin{cases} (-1)^{L(n)}, & \text{if } w(m) = w(n) \\ 0, & \text{otherwise;} \end{cases}$$

$$E_o(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

We recall that an arithmetic function $f(n)$ is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$, where $(m, n)$ denotes as usual the greatest common divisor of $m$ and $n$. It is known (Vaidyanathaswamy [6], [7, section VI]; for another proof, Gioia [3],) that every multiplicative function $f$ satisfies the identity

$$(1.1) \qquad f(mn) = \sum_{\substack{a|m \\ b|n}} f(m/a)\, f(n/b)\, f^{-1}(ab)\, C(a, b),$$

[1] Partially supported by National Science Foundation Grant No. GP 1222.

[2] This author's contribution formed a part of his Ph.D. thesis submitted to the University of Missouri in January, 1964.

Canad. Math. Bull. vol. 10, no. 1, 1967

where m and n are arbitrary positive integers and $f^{-1}$ is the Dirichlet inverse of f defined by

$$\sum_{d|n} f(d)f^{-1}(n/d) = E_o(n) .$$

We give here a generalization of this identity which holds in the case of <u>generalized Dirichlet products of arithmetic functions</u> introduced by the authors [5]. We also obtain another identity valid in the case of unitary products.

2. <u>Preliminaries</u>. Let K(n) be a fixed arithmetic function satisfying $K(1) = 1$ and for arbitrary positive integers a, b, c,

$$K((a,b))K((ab,c)) = K((a,bc))K((b,c)) . \tag{2.1}$$

For any arithmetic functions f and g, their <u>generalized Dirichlet product</u> f.g is the arithmetic function defined by

$$(f.g)(n) = \sum_{d|n} f(d)\, g(n/d)\, K((d, n/d)) .$$

It can be verified (see [4]) that (2.1) assures the associativity of the product, and together with the condition $K(1) = 1$ it implies that the kernel K(n) is multiplicative. In the sequel we shall refer to the generalized Dirichlet product as the <u>K-product</u>. We note without proof that under the K-product operation the set of multiplicative functions forms an Abelian group G with $E_o(n)$ as the identity element. The group inverse of f in G will be denoted by $f^{-1}$.

On taking $K(n) = 1$ for all n, and $K(n) = E_o(n)$, the K-product of f and g becomes, respectively, the ordinary Dirichlet product $\sum_{d|n} f(d)\, g(n/d)$ and the unitary product

$$\sum_{\substack{d|n \\ (d,n/d)=1}} f(d)\, g(n/d) .$$

The latter of these has been studied extensively by Eckford Cohen ([1], [2]).

3. A generalized identity for the K-product. We will first note the following

LEMMA. If $(a, b) = 1$, $(a, d) = 1$, and $(b, c) = 1$ then $K((ab, cd)) = K((a, c))\, K((b, d))$.

Proof. The result follows immediately from the multiplicativity of $K$ after observing that under the hypotheses of the lemma we have $((a, c), (b, d)) = 1$ and $(a, c)(b,d) = (ab, cd)$.

COROLLARY.

$$K\Big(\Big(\prod_{i=1}^{t} p_i^{x_i}, \prod_{i=1}^{t} p_i^{y_i}\Big)\Big) = \prod_{i=1}^{t} K((p_i^{x_i}, p_i^{y_i})), \qquad x_i, y_i \geq 0 .$$

From the definition of the function $C(a, b)$, we notice that we also have

$$C\Big(\prod_{i=1}^{t} p_i^{x_i}, \prod_{i=1}^{t} p_i^{y_i}\Big) = \prod_{i=1}^{t} C(p_i^{x_i}, p_i^{y_i}), \qquad x_i, y_i \geq 0 . \tag{3.1}$$

We can now prove

THEOREM 1. For arbitrary positive integers $m$ and $n$, every multiplicative function $f$ satisfies the identity

$$f(mn) = \sum_{\substack{a \mid m \\ b \mid n}} f(m/a)\, f(n/b)\, f^{-1}(ab)\, K((mn/ab, ab))\, K((m/a, n/b))\, C(a, b) .$$

Proof. Define the function

$$S(m, n) = \sum_{\substack{a \mid m \\ b \mid n}} f(m/a)f(n/b)f^{-1}(ab)K((mn/ab, ab))K((m/a,n/b))C(a,b) .$$

We shall show that $S(m, n) = f(mn)$ for all $m, n \geq 1$. First, let $m = p^x$ and $n = p^y$, where $p$ is prime and $x, y \geq 1$. Then

$$S(m,n) = f(m)f(n)K((m,n))$$

$$- \sum_{\substack{a|m \\ a>1}} f(\frac{m}{a})K((\frac{m}{a}, na)) \sum_{\substack{b|n \\ b>1}} f(\frac{n}{b})f^{-1}(ab)K((\frac{n}{b}, ab)) ,$$

where we have used equation (2.1).

Since $a|p^x$,

$$0 = E_o(na) = \sum_{b|a} f(nb)\, f^{-1}(a/b)\, K((nb, a/b))$$

$$+ \sum_{\substack{b|n \\ b>1}} f(n/b)\, f^{-1}(ab)\, K((n/b, ab)) ,$$

so that

$$S(m,n) = f(m)f(n)K((m,n))$$

$$+ \sum_{\substack{a|m \\ a>1}} f(\frac{m}{a})K((\frac{m}{a}, na)) \sum_{b|a} f(nb)f^{-1}(\frac{a}{b})K((nb, \frac{a}{b}))$$

$$= \sum_{a|m} \sum_{b|a} f(m/a)f(nb)f^{-1}(a/b)K((nb, a/b))K((na, m/a)) .$$

Interchanging the order of summation and using (2.1) again,

$$S(m,n) = \sum_{b|m} f(nb)K((nb, m/b)) \sum_{\substack{a|m \\ b|a}} f(m/a)f^{-1}(a/b)K((a/b, m/a))$$

$$= \sum_{b|m} f(nb)\, K((nb, m/b))\, E_o(m/b) = f(mn) .$$

Furthermore, since $S(1,n) = f(n)$ and $S(m,1) = f(m)$, we see that $S(m,n) = f(mn)$ for $m = p^x$, $n = p^y$ with $x, y \geq 0$. Now from the above corollary and (3.1) we have

$$S\Big(\prod_{i=1}^{t} p_i^{x_i}, \prod_{i=1}^{t} p_i^{y_i}\Big) = \sum_{\substack{a_1 | p_1^{x_1} \\ \vdots \\ a_t | p_t^{x_t}}} \sum_{\substack{b_1 | p_1^{y_1} \\ \vdots \\ b_t | p_t^{y_t}}} f\Big(\prod \frac{p_i^{x_i}}{a_i}\Big) f\Big(\prod \frac{p_i^{y_i}}{b_i}\Big) f^{-1}\Big(\prod a_i b_i\Big)$$

$$\times K\Big(\Big(\prod \frac{p_i^{x_i}}{a_i}, \prod \frac{p_i^{y_i}}{b_i}\Big)\Big) K\Big(\Big(\prod \frac{p_i^{x_i+y_i}}{a_i b_i}, a_i b_i\Big)\Big)$$

$$\times C\Big(\prod a_i, \prod b_i\Big)$$

$$= \prod_{i=1}^{t} S(p_i^{x_i}, p_i^{y_i})$$

$$= \prod_{i=1}^{t} f(p_i^{x_i+y_i}) = f\Big(\prod p_i^{x_i+y_i}\Big),$$

and the theorem is proved.

In addition to Vaidyanathaswamy's identity (1.1), the following is another interesting special case of Theorem 1, and is kindly supplied by the referee.

Let $L$ denote the set of the integers $n$ with the property that each prime divisor of $n$ has multiplicity at least 2, and let $\lambda(n)$ denote the characteristic function of $L$. It is easily observed that $\lambda(n)$ satisfies the associativity condition, (2.1). Theorem 1 becomes now, with $f^{-1}$ representing the inverse of f with respect to the kernel $\lambda$,

$$f(mn) = \Sigma\, f(d) f(\delta) f^{-1}(ab) C(a,b)\ .$$
$$ad = m\ ,\quad b\delta = n\ .$$
$$(d,\delta) \in L$$
$$(ab,\ d\delta) \in L$$

4. An identity for unitary products. For Dirichlet products, $K((m,n)) = 1$ for all $m$ and $n$, and the identity of Theorem 1 reduces to (1.1). However, in the case of unitary products, Theorem 1 reduces to a triviality. To see this, we require the

LEMMA. If $f$ is a multiplicative function and if $f^{-1}$ denotes the unitary inverse of $f$, then $f^{-1}(n) = (-1)^{L(n)} f(n)$ for all positive integers $n$.

Proof. The result is obvious if $n = 1$. For any prime $p$ and any positive integer $x$,

$$0 = E_o(p^x) = \sum_{\substack{d \mid p^x \\ (d,\, p^x/d) = 1}} f^{-1}(d)\, f(p^x/d) = f(p^x) + f^{-1}(p^x)\ ,$$

or $f^{-1}(p^x) = (-1)^{L(p^x)} f(p^x)$. Since $f^{-1}$ and $f$ are multiplicative, the lemma follows for any $n$.

Now for the unitary product, $K((m,n)) = E_o((m,n))$ ; hence, if we write $m = m_1 m_2$ and $n = n_1 n_2$, where $w(m_1) = w(n_1)$ and $(m_1, m_2) = (n_1, n_2) = 1$ and $(m_i, n_j) = 1$ except for $i = j = 1$, we see that

$$K((mn/ab,\, ab))\ \ K((m/a,\ n/b))\ \ C(a,b)$$

vanishes unless $a = m_1$ and $b = n_1$. Using the lemma it is seen that the identity reduces to the obvious relation $f(mn) = f(m_2)\ f(m_1 n_1)\ f(n_2)$ .

We will now give a non-trivial identity for the unitary product. We write $d \| n$ to mean that $d$ is a unitary divisor of $n$, i.e. $d | n$ and $(d, n/d) = 1$. Let

$$\lambda(a,b) = \begin{cases} (-1)^{L(a)}, & \text{if } w(a) | w(b) \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 2. For arbitrary positive integers $m$ and $n$ and for any multiplicative function $f$,

$$f(mn) = \sum_{\substack{a \| m \\ b \| n \\ w(b) | w((m,n)) \\ w(a) | w((m,n))}} f(m/a)\, f(n/b)\, f^{-1}(ab)\, \lambda(a,b) .$$

<u>Proof.</u> Let

$$T(m,n) = \sum_{\substack{a \| m \\ b \| n \\ w(b) | w((m,n)) \\ w(a) | w((m,n))}} f(m/a)\, f(n/b)\, f^{-1}(ab)\, \lambda(a,b)$$

$$= \sum_{\substack{a \| m \\ b \| n \\ w(a) | w(b) | w((m,n))}} f(m/a)\, f(n/b)\, f(ab)\, (-1)^{L(a)+L(b)}$$

Clearly, $T(1,n) = f(n)$ and $T(m,1) = f(m)$ for all $m, n$. If $p$ is a prime and $x, y \geq 1$, for $m = p^x$ and $n = p^y$ we have

$$T(m,n) = f(m)\, f(n) + f(m)\, f^{-1}(n) - f^{-1}(mn) = f(mn) ,$$

using the above lemma. Therefore,

$$T\left(\prod_{i=1}^{t} p_i^{x_i}, \prod_{i=1}^{t} p_i^{y_i}\right)$$

$$= \sum_{\substack{a_1 \| p_1^{x_1} \\ \vdots \\ a_t \| p_t^{x_t} \\ w(a_1) | w(b_1) | w((p_1^{x_1}, p_1^{y_1})) \\ \vdots \\ w(a_t) | w(b_t) | w((p_t^{x_t}, p_t^{y_t}))}} \sum_{\substack{b_1 \| p_1^{y_1} \\ \vdots \\ b_t \| p_t^{y_t}}} f\left(\Pi \frac{p_i^{x_i}}{a_i}\right) f\left(\Pi \frac{p_i^{y_i}}{b_i}\right) f(\Pi\, a_i b_i)(-1)^{\Sigma[L(a_i)+L(b_i)]}$$

$$= \prod_{i=1}^{t} T(p_i^{x_i}, p_i^{y_i})$$

$$= \prod_{i=1}^{t} f(p_i^{x_i+y_i}) = f(\Pi\, p_i^{x_i+y_i}) .$$

A restatement of theorem 2 would be as follows:

If $f$ is multiplicative, then for arbitrary integers $m, n$,

$$f(mn) = \sum_{\substack{a \| m \\ b \| n \\ w(a) | w(b) | w((m,n))}} f(m/a)\, f(n/b)\, f(ab)\, (-1)^{L(a)+L(b)} .$$

## REFERENCES


1. Eckford Cohen, Arithmetical functions associated with the

unitary divisors of an integer. Math. Zeit., 74 (1960), pages 66-80.

2. Eckford Cohen, Unitary functions (mod r). Duke Math. J., 28 (1961), pages 475 - 485.

3. A. A. Gioia, On an identity for multiplicative functions. Amer. Math. Monthly, 69 (1962), pages 988 - 991.

4. A. A. Gioia, The K-product of arithmetic functions. Can. J. Math., 17 (1965), pages 970-976.

5. A. A. Gioia and M. V. Subbarao, Generalized Dirichlet products of arithmetic functions (Abstract). Notices Amer. Math. Soc., 9 (1962), page 305.

6. R. Vaidyanathaswamy, The identical equation of the multiplicative functions. Bull. Amer. Math. Soc., 36 (1930), pages 762-772.

7. R. Vaidyanathaswamy, The theory of multiplicative arithmetic functions. Trans. Amer. Math. Soc., 33 (1931) pages 579-662.

University of Alberta,
University of Kerala and
Texas Technological College