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SOME CLASSES OF COMPLETE SEQUENCES AND APPROXIMATIONS IN NORMED LINEAR SPACES (1)

DEDICATED TO THE MEMORY OF MY REVERED TEACHER, K. ANANDA RAU

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1. **Introduction.** Let X denote a real or complex normed linear space, and X^* its conjugate space. The closed linear manifold V of X is said to be spanned or generated by the sequence $\{f_n\}$ of elements of X if every g in V is the limit of a sequence of linear combinations of the elements of $\{f_n\}$. This means that to every arbitrary $\epsilon > 0$ there corresponds a finite integer $n = n(\epsilon)$ and a set of scalars $a_{\nu s}$ such that

$$\left\| g - \sum_{\nu=1}^{n} a_{\nu n} f_\nu \right\| < \epsilon.$$

It is well known that a necessary and sufficient condition that a sequence $\{f_n\}$ of vectors span the closed linear manifold V of X is that every ϕ in X^* which is orthogonal to every f_n is orthogonal to the entire manifold V. This result furnishes the essential tool for obtaining the so-called closure theorems for various normed linear spaces. If the sequence of elements $\{f_n\}$ in X spans the whole space X, it is said to be 'complete' (also called sometimes 'total' or 'closed'). Thus the sequence $\{f_n\}$ is complete if and only if, for any ϕ in X^*, the relation $\phi(f_n) = 0$ $(n = 1, 2, \ldots)$ implies $\phi = 0$.

Philip Davis and Ky Fan [1] introduced the following special classes of complete sequences.

(i) '$\{a_n\}$-complete sequences': Given a sequence $\{a_n\}$ of non-negative numbers (real or complex according as X is real or complex) a sequence $\{f_n\}$ of elements in X is said to be $\{a_n\}$-complete if, for a ϕ in X^*, $\phi(f_n) \leqslant a_n$ $(n = 1, 2, \ldots)$ implies $\phi = 0$.

(ii) Given $p \geqslant 1$, $\{f_n\}$ is said to be complete of order p if, for a ϕ in X^*, the convergence of the series $\Sigma |\phi(f_n)|^p$ implies $\phi = 0$.

In [6] Suryanarayana introduced and studied other classes of complete sequences.

Davis and Fan obtained approximation properties which characterize these classes of complete sequences. For example, they showed ([1], Theorem 1) that a sequence of elements in X is $\{a_n\}$-complete if and only

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if, for any element g in X and for any positive number ϵ, there exists a finite number of coefficients $c_1, c_2, \ldots, c_m$ such that

$$\| g - c_n f_n \| < \epsilon; \quad \sum_{n=1}^{m} |c_n| a_n < \epsilon.$$

Thus the $\{a_n\}$-complete sequences are examples of classes of complete sequences which can be obtained by imposing some restriction on the coefficients $c_1, c_2, \ldots, c_m$, like the second inequality above.

In the first few sections of this paper we shall introduce, using this method of imposing a restriction on the coefficients, a very general class of complete sequences (called E-complete sequences) and characterize it in terms of linear continuous functionals on X. The classes of complete sequences considered by Philip Davis and Ky Fan and Suryanarayana are shown to be special cases of our class.

2. ***E*-completeness.** Throughout what follows X is a normed linear space with norm $\| \;\; \|$, E a locally convex linear topological metric space with a countable basis $\{u_n\}$ $(n = 1, 2, \ldots)$ and with metric $d(x, y)$ satisfying $d(x, y) = d(x-y, 0)$. We write $\rho(x-y)$ for $d(x-y, 0)$. In particular E may be a normed linear space with a countable basis. Its norm will then be denoted by $\| \;\; \|_E$. E^* as usual denotes the conjugate of E.

DEFINITION. A sequence $\{f_n\}$ in X is said to be 'E-complete' if, given any $\epsilon > 0$ and any g in X, there exist coefficients $c_1, c_2, \ldots, c_m$ such that

$$\left\| g - \sum_{n=1}^{m} c_n f_n \right\| < \epsilon, \quad \rho\left(\sum_{n=1}^{m} c_n u_n \right) < \epsilon. \qquad \ldots \qquad \ldots \qquad (2.1)$$

We have the following basic result suggested by [2], Theorem 2.

THEOREM 1. *A sequence $\{f_n\}$ is E-complete if and only if the following property holds:*

Given a ϕ in X^*, there exists a ψ in E^* satisfying $\psi(u_n) = -\phi(f_n)$ $(n = 1, 2, 3 \ldots)$ if and only if $\phi = 0$. $\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$

Proof. Consider the direct sum space $X \oplus E$ of elements of the form $\{g, v\}$ with g and v in X and E respectively, and with the invariant metric $\rho_1(\{g, v\}, 0) = \| g \| + \rho(v)$. Let X_1 denote the closed linear subspace of $X \oplus E$ formed by all pairs of the form $\{g, 0\}$. Let Y be the closed linear subspace of $X \oplus E$ spanned by the sequence of elements $f_n' = \{f_n, u_n\}$. The property (2.1) holds if and only if $Y \supset X_1$. The latter is equivalent to the property that every linear continuous functional on $X \oplus E$ which vanishes on Y must also vanish on X_1; that is, for ϕ in X and ψ in E, the equations $\phi(f_n) + \psi(u_n) = 0$ $(n = 1, 2, \ldots)$ imply $\phi = 0$. This is easily seen to be equivalent to property (2.2).

As will be shown in detail in the next section, the $\{a_n\}$-complete sequences, and sequences complete of order p introduced in [1] are special cases of E-complete sequences. One can see that Theorems 1 and 2 proved in [1] are indeed special cases of our Theorem 1. In fact, many of the results in [1] can be generalized with appropriate changes for E-complete sequences. As an example, we mention the following.

THEOREM 2. *Suppose* $\{f_n\}$ *and* $\{g_n\}$ *are two sequences of elements in* X, *and* λ *is a number satisfying* $0 \leqslant \lambda < 1$, *and*

$$\left\| \sum_{n=1}^{m} c_n(f_n - g_n) \right\| \leqslant \lambda \left\| \sum_{n=1}^{m} c_n f_n \right\|$$

for any finite set of coefficients $c_1, c_2, \ldots, c_m$. *Then, if* $\{f_n\}$ *is* E-*complete, so also is* $\{g_n\}$.

3. **Special cases.**

3.1. Taking $E = l_q$, $q > 1$, we get completeness of order p $\left(\text{where } \frac{1}{p} + \frac{1}{q} = 1\right)$ (Theorem 2 of [2]).

3.2. Taking E to be the space of elements $x = (x_1, x_2, \ldots)$ where $x_1, x_2, \ldots$ are real or complex numbers for which

$$\|x\|_E = \sum_{i=1}^{\infty} a_i |x_i| < \infty,$$

where $\{a_i\}$ $(i = 1, 2, \ldots)$ is a given sequence of positive numbers, we see that E^* consists of sequences $(y_1, y_2, \ldots)$ of complex numbers for which $|y_n|/a_n$ is bounded $(n = 1, 2, \ldots)$. We now obtain $\{a_n\}$-completeness ([2], Definition 1, Theorem 1).

3.3. Let E be the normed linear space of all sequences $x = (x_1, x_2, \ldots)$ for which, for a given $r > 0$,

$$\|x\|_E = \sum_{n=1}^{\infty} |x_n| n^r < \infty.$$

Then the corresponding completeness is the E_r-completeness of [6], Definition 2.

3.4. If E is the normed linear space of sequences $x = (x_1, x_2, \ldots)$ for which

$$\|x\|_E = \underset{n}{\text{l.u.b.}}\, |x_n|/n^r < \infty$$

for a given $r > 0$, we get F_r-completeness ([6], Definition 3).

3.5. We obtain R-completeness $(R > 0)$ ([6], Definition 4) on taking E to be the space of elements $x = (x_1, x_2, \ldots)$ for which

$$\|x\|_E = \underset{n}{\text{l.u.b.}}\, |x_n|/R^n < \infty.$$

3.6. Let M be a given linear manifold and $[M, d_i]$ a sequence of locally convex linear topological spaces metrized by $d_i(x, y)$ $(i = 1, 2, \ldots)$ such that any two members of the sequence have comparable topologies.(2) Let $[M, d]$ denote the l.u.b. topology of the sequence $[M, d_i]$ and $[M, d_i]^*$, $[M, d]^*$ respectively the set of linear continuous functionals of the spaces $[M, d_i]$ and $[M, d]$. It is known that $[M, d]$ is metrizable by the usual sigma metric $d(x, y)$ where $d(x, y) = d(x-y, 0)$ is given by $d(x, 0) = \sum_{i=1}^{\infty} F(d_i(x))/2^i$, where $F(u) = u/(1+u)$. Also ([3], [6]) we have $[M, d]^* = \bigcup_i [M, d_i]^*$. Let (U_n) $(n = 1, 2, \ldots)$ be a basis for $[M, d]$; it is then a basis for $[M, d_i]$ for all i. Theorem 1 now shows that

3.6.1. *A sequence $\{f_n\}$ is $[M, d]$-complete in X if and only if, for a given ϕ in X^*, there exists a ψ in E^* with $\psi(u_n) = -\phi(f_n)$ $(n = 1, 2, \ldots)$ if and only if $\phi = 0$.*

We shall give two simple applications of this result.

3.7. *l_{p+}-completeness.* A sequence $\{f_n\}$ in X is defined to be l_{p+}-complete $(p > 1)$ if, given any $\epsilon > 0$ and any g in X, there exists a finite number of coefficients $c_1, \ldots, c_m$ such that

$$\left\| g - \sum_{n=1}^{m} c_n f_n \right\| < \epsilon; \quad (\Sigma |c_n|^q)^{\frac{1}{q}} < \epsilon$$

for every $q > p$.

Applying the result 3.6.1 we see that

3.7.1. *$\{f_n\}$ is l_{p+}-complete if and only if, for a ϕ in X^* for which $\Sigma |\phi(f_n)|^{p-\epsilon} < \infty$ for some $\epsilon > 0$, we have $\phi = 0$.*

3.8. *Γ-complete sequences.* Using the notation and results obtained in [2], we are led to the class of complete sequences given by the following

DEFINITION. A sequence $\{f_n\}$ $(n = 0, 1, 2, \ldots)$ in X is said to be Γ-complete if, given any g in X and any positive ϵ, there exists a finite number of coefficients $c_0, c_1, \ldots, c_m$ such that

$$\left\| g - \sum_{n=0}^{m} c_n f_n \right\| < \epsilon; \quad \text{l.u.b.} \{|c_0|, |c_n|^{1/n} \ (n = 1, \ldots, m)\} < \epsilon.$$

Applying the result 3.6.1 and the theorems obtained in [2], we obtain the following.

THEOREM 3. *A necessary and sufficient condition for $\{f_n\}$ to be Γ-complete is that the only ϕ in X^* for which* $\text{l.u.b.}_n |\phi(f_n)|/R^n < \infty$ *for some $R > 0$ is $\phi = 0$.*

4. **c_0-completeness and E_{0+}-completeness.** There are some other types of complete sequences which are better studied directly than as special cases of E-completeness.

(2) Two topologies are said to be comparable if one of them is finer (also called weaker sometimes), i.e. its ring of open sets contains that of the other. The l.u.b. topology of a given family of topologies defined on M is the finest of the family.

One of them is c_0-completeness introduced in [6] and defined as follows. A sequence $\{f_n\}$ of elements in a normed linear space X is said to be c_0-complete if $\phi = 0$ is the only member of X^* for which $\{\phi(f_n)\}$ is a null sequence.

Some interesting properties of this class of complete sequences are obtained in [6]. We shall introduce here a sub-class of c_0-complete sequences, and obtain some results concerning the same.

4.1. DEFINITION. A sequence $\{f_n\}$ of elements in a space X is said to be E_{0+}-complete if $\phi = 0$ is the only member of X^* for which $\{\phi(f_n)/n^a\}$ is a null sequence for every $a > 0$.

It is obvious that a sequence $\{f_n\}$ which is E_{0+}-complete is also c_0-complete (and hence complete in the usual sense). The following example shows that c_0-completeness need not always imply E_{0+}-completeness.

Taking X to be the space l_p $(p > 1)$, consider the sequence of elements $f_n = \sum_{d|n} u_d/d$ $(n = 1, 2, \ldots)$ where $u_n = (\delta_{in})$ where δ_{in} is the Kronecker δ. The sequence $\{f_n\}$ is c_0-complete in l_p as can be seen, for example, by using [6], Theorem 2. But it is not E_{0+}-complete, since for $\phi = (1, 0, 0, \ldots)$ in l_p^* we have $\phi(f_n) = 1$ $(n = 1, 2, \ldots)$. Thus, $\{\phi(f_n)/n^a\}$ is a null sequence for every $a > 0$ while $\phi \neq 0$.

The construction of E_{0+}-complete sequences in a normed vector space X is no more difficult than that of c_0-complete sequences. For example, we have the following simple result.

4.2. *If $\{f_n\}$ is c_0-complete, then, for any $a > 0$, $\{n^a f_n\}$ is E_{0+}-complete.*

This follows very easily, since, if for a ϕ in X^*, the sequence $\{\phi(n^a f_n)/n^b\}$ is null for every $b > 0$, then $\{\phi(f_n)\}$ must be null (on taking $b = a$) and hence $\phi = 0$. A less trivial result is

THEOREM 4. *Let $\{f_n\}$ be a c_0-complete sequence, with $\|f_n\| = 1$, in a normed linear space X. Let a, k be any (real or complex) numbers $\neq 0$. Let $\{b_n\}$ be any bounded sequence of numbers. Then the sequence $\{g_n\}$ given by $g_n = k\, n^a f_n + \sum_{d|n} f_d\, b_{n/d}$ $(n = 1, 2, \ldots)$ is E_{0+}-complete in X.*

Proof. It is sufficient to consider ϕ in X^* for which $\|\phi\| \leqslant 1$. Let $\{\phi(g_n)/n^e\}$ be a null sequence for every $e > 0$. If $t(n)$ denotes the number of divisors of n, it is well known that $\{t(n)/n^e\}$ is a null sequence for every $e > 0$. Hence, if $B = \operatorname{l.u.b.}_n b_n < \infty$, we have

$$\phi\Big(\sum_{d|n} f_d\, b_{n/d}\Big)\Big/ n^e \leqslant D\, \|\phi\|\, \Sigma\, \|f_d\|/n^e$$

$$\leqslant B \sum_{d|n} 1/n^e = B\, t(n)/n^e$$

$$\to 0 \quad \text{as } n \to \infty \quad \text{for every } e > 0.$$

Hence, if $\{\phi(g_n)/n^e\}$ is a null sequence for every $e > 0$, so also is $\{\phi(n^a f_n)/n^e\}$. Hence $\{\phi(f_n)\}$ is null and hence $\phi = 0$, completing the proof of the theorem.

The following theorem gives a necessary and sufficient condition for E_{0+}-completeness in a normed linear space X.

THEOREM 5. *A given sequence $\{f_n\}$ of elements in X is E_{0+}-complete if and only if, for any g in X, any $\epsilon > 0$, any $e > 0$ and any null sequence $\{a_n\}$, there exists a finite number of coefficients $c_1, c_2, \ldots, c_m$ (real or complex depending on X) such that*

$$\left\| g - \sum_{n=1}^{m} c_n f_n \right\| < \epsilon; \quad \sum_{n=1}^{m} |c_n|\, |a_n|\, n^e < \epsilon.$$

This follows by an application of ([6], Theorem 1) or ([1], Theorem 1).

5. **Construction of c_0-complete sequences.** The following two theorems demonstrate some methods which can be used to construct c_0-complete sequences. The first of these (Theorem 6) is a wide generalization of ([6], Theorem 2) and ([6], p. 328, Corollary).

THEOREM 6. *Let $\{g_n\}$ be a complete sequence in X, such that the sequence $\{\| g_n \|\}$ is bounded. Let $\{a_n\}$ be a null sequence with non-zero terms. Let $k(n)$ be any (real or complex valued) bounded arithmetic function for which $k(1) \neq 0$. Let r be any integer $\geqslant 1$. For $n = 1, 2, \ldots$, let $f_n = \sum_{d^r \mid n} a_d\, g_d\, k((d^r, n/d^r))$ where (m, n) denotes as usual the greatest common divisor of m and n. Then $\{f_n\}$ is c_0-complete in X.*

Proof. We proceed as in the proof of [6], Theorem 2. Let $\{\phi(f_n)\}$ be a null sequence with ϕ in X^*. It is sufficient to take only those ϕ for which $\| \phi \| \leqslant 1$. Since $f_{p_n^r} = a_1 g_1 k(1) + a_{p_n} g_{p_n} k(1)$, we have

$$k(1)\, |a_1|\, |\phi(g_1)| \leqslant \left| \phi\left(f_{p_n^r}\right) \right| + |a_{p_n}|\, k(1)\, \|\phi\|\ \|g_{p_n}\|$$
$$\to 0 \ \text{ as } n \to \infty,$$

since $\| g_{p_n} \|$ is bounded and $\{a_{p_n}\}$ and $\left\{\phi\left(f_{p_n^r}\right)\right\}$ are null sequences. Thus $\phi(g_1) = 0$. We now proceed by induction and assume that $\phi(g_i) = 0$ $(i = 1, 2, \ldots, m-1)$. Take $p_n > m$ and consider the null subsequence

$$\phi\left(f_{m^r p_n^r}\right).$$

We have

$$\phi\left(f_{m^r p_n^r}\right) = \sum_{d \mid m} a_d\, \phi(g_d)\, k((d^r, m^r p_n^r / d^r))$$
$$+ a_m\, \phi(g_m)\, k(1) + \sum_{d \mid m} a_{dp_n}\, \phi(g_{dp_n})\, k((p_n^r d^r, m^r/d^n))$$
$$= a_m\, \phi(g_m)\, k(1) + \sum_{d \mid m} a_{dp_n}\, \phi(g_{dp_n})\, k((d^r, m^r/d^r)).$$

As $n \to \infty$, the left member and each term in the summation in the right member tend to zero, giving $\phi(g_m) = 0$. Thus, $\phi(g_n) = 0$ for all n and hence $\phi = 0$, completing the proof.

THEOREM 7. *Let $\{a_n\}$ be a null sequence of non-zero numbers and $\{b_n\}$ a bounded sequence for which the subsequence $\{b_{p_n}\}$ (where p_n is the nth prime) converges to a limit $\neq -b_1$. Let $\{g_n\}$ be a complete sequence in X. Define*

$$f_n = \sum_{d_1 d_2 | n} a_{d_1} g_{d_1} b_{d_2} \quad (n = 1, 2, \ldots).$$

Then $\{f_n\}$ is c_0-complete.

Proof. Let for a ϕ in X^*, $\|\phi\| \leqslant 1$ and $\{\phi(f_n)\}$ be null. Since

$$f_{p_n} = a_1 g_1(b_1 + b_{p_n}) + a_{p_n} g_{p_n} b_1$$

we get, on equating the values of ϕ at the elements on either side of this relation and letting $n \to \infty$, that $\phi(g_1) = 0$. As before, we shall proceed by induction and assume that $\phi(g_1) = \ldots = \phi(g_{m-1}) = 0$. Taking $p_n > m$,

$$f_{mp_n} = \sum_{d|m} a_d g_d \sum_{e|(mp_n/d)} b_e + a_m g_m \sum_{e|p_n} b_e + \sum_{d|m} a_{dp_n} g_{dp_n} \sum_{e|(m/d)} b_e.$$

The induction hypothesis gives

$$\phi(f_{mp_n}) = a_m \phi(g_m)(b_1 + b_{p_n}) + \sum_{d|m} a_{dp_n} \phi(g_{dp_n}) \sum_{e|(m/d)} b_e.$$

Since the left member and every term in the summation on the right separately tend to zero as $n \to \infty$, we obtain

$$a_m \phi(g_m)(b_1 + \lim b_{p_n}) = 0$$

giving $\phi(g_m) = 0$, thus completing the proof.

6. **Examples.** First we introduce some arithmetic functions.
Let $n = p_1^{a_1} \ldots p_k^{a_k}$ be the prime power representation for $n > 1$. Let $\sigma_k(n)$, $\sigma_k^*(n)$, $\sigma_k^{**}(n)$ and $J_k(n)$ be defined by

$$\sigma_k(1) = \sigma_k^*(1) = \sigma_k^{**}(1) = J_k(1) = 1,$$

and for $n > 1$ with the above factorization into primes,

$$\sigma_k(n) = \prod_i (1 + p_i^k + \ldots + p_i^{ka_i}),$$

$$\sigma_k^*(n) = \prod_i (1 + p_i^{ka_i}),$$

$$\sigma_k^{**}(n) = \prod_i w(p_i^{a_i})$$

where

$$w(p^a) = \begin{cases} 1 + p^k + p^{2k} + \ldots + p^{ak}, & 0 < a < 4, \\ 1 + p^k + p^{(a-1)k} + p^{ak}, & a \geqslant 4, \end{cases}$$

and

$$J_k(n) = n^k \prod_i \left(1 - \frac{1}{p_i^k}\right).$$

The arithmetic interpretations for these functions are that $\sigma_k(n)$ represents the sum of the kth powers of the divisors of n; $\sigma_k^*(n)$ represents the sum of the kth powers of the 'unitary' divisors of n, namely those divisors d of n which are prime to their conjugates; $\sigma_k^{**}(n)$ represents the sum of the kth powers of those divisors d of n whose greatest common divisor with the corresponding conjugate divisor n/d is square-free. Finally, if k is a positive integer, $J_k(n)$ is the Jordan's generalization of Euler's totient function, which stands for the number of ordered sets of k positive integers whose greatest divisor is prime to n, or, alternately, $J_k(n)$ represents the number of integers in a complete residue system modulo n^k whose greatest common divisor with n^k is k-power free (ignoring the trivial k-power 1).

Let $\Gamma(R)$, $R > 0$, denote the normed linear space of all power series $\alpha(z)$ given by

$$\alpha(z) = \sum_{n=0}^{\infty} a_n z^n$$

for which $N(\alpha; R) = \sum_{n=0}^{\infty} |a_n| R^n < \infty$, with $N(\alpha; R)$ as the norm of $\alpha(z)$. This space is easily seen to be complete and its dual $\Gamma^*(R)$ consists of functionals $\phi = (b_0, b_1, b_2, \ldots)$ such that $\text{l.u.b.}_i\, |b_i|/R^i < \infty$, and $\phi(\alpha(z)) = \Sigma\, b_i a_i$. Convergence in $\Gamma(R)$ is uniform convergence over the circle $|z| = R$.

It is known that the sequence $\alpha_n(z)$ given by

$$\alpha_n(z) = 1 + \sum_{p=0}^{\infty} z^p \left(1 - \frac{1}{n}\right)^p \Big/ p^2 \quad (n = 1, 2, \ldots)$$

is a complete sequence in $\Gamma(1)$ ([4], Theorem 3).

Let $s > 0$ be fixed and let $h_s(n)$ denote any one of the functions $\sigma_s(n)$, $\sigma_s^*(n)$, $\sigma_s^{**}(n)$ and $J_s(n)$. For $n = 1, 2, \ldots$, we define

$$f_n = a_{n,1} + \sum_{p=1}^{\infty} a_{n,p+1} z^p/p^2 \tag{6.1}$$

where, for each n ($n = 1, 2, \ldots$) and each p ($p = 0, 1, 2, \ldots$), we let

$$a_{n,p+1} = \sum_{i=0}^{p} (-1)^i \binom{p}{i} h_{s+i}(n)/n^i,$$

$\binom{p}{i}$ being the usual binomial coefficient and $\binom{0}{0}$ defined to be unity.

We then have the result that

6.2 *The sequence* $\{n^t f_n\}$ *is* E_{0+}*-complete in* $\Gamma(1)$ *for all* $t > -s$.

To show this, we use Theorem 6 setting, in its statement, $r = 1$, $g_n = \alpha_n$ and

(i) $a_n = \mu(n)/n^s$, $k(n) = 1$ in the case $h_s(n) = J_s(n)$

(ii) $a_n = 1/n^s$, $k(n) = 1$ if $h_s(n) = \sigma_s(n)$

(iii) $a_n = 1/n^s$, $k(n) = [1/n]$ if $h_s(n) = \sigma_s^*(n)$

(iv) $a_n = 1/n^s$, $k(n) = |\mu(n)|$ if $h_s(n) = \sigma_s^{**}(n)$.

Here $\mu(n)$ stands for the well-known Möbius arithmetic function and $[a]$ denotes as usual the largest integer not exceeding a.

We observe that the sequence $\{\alpha_n\}$ is bounded in norm since

$$N(\alpha_n, 1) = 1+\sum_{p=1}^{\infty}\left(1-\frac{1}{n}\right)^p \Big/ p^2$$
$$< 1+\Sigma\, 1/p^2 = 1+\pi^2/6.$$

From Theorem 6 it follows that the sequence $\{\beta_n\} = \{\beta_n(z)\}$ is c_0-complete in $\Gamma(1)$, where

$$\beta_n = \sum_{d\mid n} a_d\,\alpha_d\,k((d, n/d)) \quad (n = 1, 2, \ldots)$$
$$= f_n/n^s$$

where f_n is given by (6.1).

Applying the result 4.2 for the sequence $\{f_n/n^s\}$, we obtain 6.2.

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