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# A FAMILY OF COMBINATORIAL IDENTITIES

BY

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*In memory of Leo Moser*

1. **Introduction.** In a recent paper, Murray Eden [5] generalized the simple identity for the Eulerian product,

$$1+\sum_{n=0}^{\infty} bx^{n+1}\prod_{i=1}^{n}(1+bx^i) = \prod_{i=1}^{\infty}(1+bx^i), \tag{1.1}$$

and obtained the following infinite family of identities:
For $h=1, 2, 3, \ldots$, let

$$F_h(b; x) = \sum_{n=0}^{\infty} b^h x^{h(n+1)}\prod_{i=1}^{n}(1+bx^i), \tag{1.2}$$

where we assume throughout that $|x|<1$, empty products equal unity and empty sums equal zero; then

$$F_h(b; x) = \prod_{j=1}^{h-1}(x^{-j}-1)\prod_{i=1}^{\infty}(1+bx^i)-1-\sum_{i=1}^{h-1} b^i \prod_{j=i+1}^{h-1}(x^{-j}-1). \tag{1.3}$$

As Eden noted, $F_h(b; x)$ is the generating function of $p_h(m, n)$ which denotes the number of partitions of $n$ into $m$ parts, in which the largest part appears exactly $h$ times and all other parts are distinct:

$$F_h(b; x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_h(m, n)b^m x^n. \tag{1.4}$$

One of our objectives in this paper is to establish an infinite family of identities (see Theorem 1 below) for the reciprocal of the Euler product,

$$\prod_{n=1}^{\infty}(1-bx^n)^{-1},$$

analogous to Eden's (1.3) for the Euler product. This we do by generalizing the simple identity

$$1+\sum_{n=1}^{\infty}\frac{bx^n}{(1-bx)(1-bx^2)\ldots(1-bx^n)} = \prod_{n=1}^{\infty}(1-bx^n)^{-1}. \tag{1.5}$$

Later in the paper, we use Eden's identities (1.3) to obtain (in Theorem 2) an infinite class of identities for $\prod_{n=1}^{\infty}(1-x^n)^{-1}$. These identities are apparently new and are not covered by the result of Theorem 1. In §3 we comment on some other

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known classes of expansions for this product. Towards the end of the paper, we consider an identity for

$$\prod_{n=1}^{\infty} (1-ax^n)(1-bx^n)^{-1},$$

of which (1.1) and (1.5) are special cases. We show that this identity can be derived from Heine's fundamental transformation for ${}_2\phi_1$, and also we give a purely combinatorial proof. We give an application of our identities proved in Theorem 2 as a conclusion to the paper.

2. THEOREM 1. *Let*

$$G_h(b; x) = \sum_{n=1}^{\infty} b^h x^{hn} \prod_{m=1}^{n} (1-bx^m)^{-1}. \tag{2.1}$$

*Then for* $h=1, 2, 3, \ldots,$

$$\begin{aligned} G_h(b; x) = {} & \prod_{j=1}^{h-1} (1-x^j) \prod_{i=1}^{\infty} (1-bx^i)^{-1} - \prod_{i=1}^{h-1} (1-x^i) \\ & - \sum_{i=1}^{h-1} b^i x^i \prod_{j=i+1}^{h-1} (1-x^j). \end{aligned} \tag{2.2}$$

**Proof.** We first note that $G_h(b; x)$ is the generating function for the number $p^{(h)}(n)$ of partitions of $n$ in which the largest part appears at least $h$ times:

$$G_h(b; x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p^{(h)}(m, n) b^m x^n, \tag{2.3}$$

where $p^{(h)}(m, n)$ denotes the number of partitions of $n$ into exactly $m$ parts (repetitions allowed) in which the largest part appears at least $h$ times. This gives, incidentally, the obvious relation

$$p^{(h)}(n) = \sum_{m=1}^{\infty} p^{(h)}(m, n). \tag{2.4}$$

Since both $G_1(b, x)$ and $\prod_{n=1}^{\infty} (1-bx^n)^{-1}$ generate the function $p^{(1)}(m, n)$ for $n>0$, [in addition to $G_1(b; x)$], we have

$$1+G_1(b; x) = \prod_{n=1}^{\infty} (1-bx^n)^{-1}. \tag{2.5}$$

This is the same as the expansion in (1.5).

We next prove the relation

$$G_h(b; x) = x^h G_h(b; x) + G_{h+1}(b; x) + b^h x^h. \tag{2.6}$$

This can be proved by using combinatorial arguments analogous to Eden's for his formula (3). But probably the simplest proof is the following one using elementary manipulation of series.

$$
\begin{aligned}
G_h(b;x)-G_{h+1}(b;x) &= \sum_{n=1}^{\infty}(b^h x^{hn}-b^{h+1}x^{(h+1)n})\prod_{m=1}^{n}(1-bx^m)^{-1}\\
&= \sum_{n=1}^{\infty}(1-bx^n)b^h x^{hn}\prod_{m=1}^{n}(1-bx^m)^{-1}\\
&= \sum_{n=1}^{\infty}b^h x^{hn}\prod_{m=1}^{n-1}(1-bx^m)^{-1}\\
&= x^h\sum_{n=0}^{\infty}b^h x^{hn}\prod_{m=1}^{n}(1-bx^m)^{-1}\\
&= x^h\{G_h(b;x)+b^h\},
\end{aligned}
$$

from which (2.6) follows at once.

Finally, we prove (2.2) using mathematical induction on $h$. In view of (2.5), (2.2) holds for $h=1$.

Suppose now that the formula holds for $h=n$. Then

$$
\begin{aligned}
G_{n+1}(b;x) &= (1-x^n)G_n(b;x)-x^n b^n\\
&= -\prod_{i=1}^{n}(1-x^i)+\prod_{i=1}^{n}(1-x^i)\prod_{k=1}^{\infty}(1-bx^k)^{-1}\\
&\quad -\sum_{i=1}^{n-1}b^i x^i\prod_{k=i+1}^{n}(1-x^k)-b^n x^n.
\end{aligned}
$$

Since the last two terms on the right side of the above equation can be together replaced by

$$
-\sum_{i=1}^{n}b^i x^i\prod_{k=i+1}^{n}(1-x^k),
$$

we see that (the formula) (2.2) holds for $G_{n+1}(b;x)$ also, thus completing the proof of the theorem.

3. **A new family of identities for $\prod_{n=1}^{\infty}(1-x^n)^{-1}$.** From (1.3) and (2.2) we get the following expansions. For $h=1,2,3,\ldots,$

$$
\begin{aligned}
\prod_{i=1}^{\infty}(1+bx^i) &= \prod_{j=1}^{h-1}(x^{-j}-1)^{-1}\{1+\sum_{n=0}^{\infty}b^h x^{h(n+1)}\prod_{i=1}^{n}(1+bx^i)\}\\
&\quad +\sum_{i=1}^{h-1}b^i\prod_{j=1}^{i}(x^{-j}-1)^{-1};
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\prod_{i=1}^{\infty}(1-bx^i)^{-1} &= 1+\sum_{i=1}^{h-1}b^i x^i\prod_{j=1}^{i}(1-x^j)^{-1}\\
&\quad +\prod_{j=1}^{h-1}(1-x^j)\sum_{n=1}^{\infty}b^h x^{hn}\prod_{m=1}^{n}(1-bx^m)^{-1}.
\end{aligned}
\tag{3.2}
$$

In particular, when $b=1$, (3.2) gives

$$\prod_{i=1}^{\infty}(1-x^i)^{-1} = 1+\sum_{i=1}^{h-1}\frac{x^i}{(1-x)\dots(1-x^i)} + \prod_{j=1}^{h-1}(1-x^j)\sum_{n=1}^{\infty}\frac{x^{hn}}{(1-x)\dots(1-x^n)}. \tag{3.3}$$

This class of expansions for $\prod_{i=1}^{\infty}(1-x^i)^{-1}$ does not seem to have been noted before. We now obtain still another class of expansions for the same product.

THEOREM 2. *For $h=1, 2, 3, \dots$, we have*

$$\prod_{n=1}^{\infty}(1-x^n)^{-1} = D^2\left\{1+\sum_{i=1}^{h-1}x^{-i}\prod_{j=i+1}^{h-1}(x^{-j}-1)^2 + \sum_{k=h}^{\infty}\left(\sum_{n=0}^{\infty}x^{h(n+1/2)}C_{k-h,n}(x)\right)^2\right\}, \tag{3.4}$$

*where*

$$D = D(x) = \prod_{j=1}^{h-1}(x^{-j}-1)^{-1} \tag{3.5}$$

*and*

$$C_{m,n}(x) = \textit{the coefficient of } b^m \textit{ in } \prod_{i=1}^{n}(1+bx^{i-1/2}). \tag{3.6}$$

**Proof.** The familiar Jacobi triple product identity gives

$$\prod_{n=1}^{\infty}(1-x^n)^{-1}\sum_{n=-\infty}^{\infty}b^n x^{n^2/2} = \prod_{n=1}^{\infty}(1+x^{n-1/2}b)\prod_{n=1}^{\infty}(1+x^{n-1/2}b^{-1}). \tag{3.7}$$

Substituting for the products on the right side by applying the formula (3.1) and then equating the coefficients of $b^k$ $(k=0, 1, 2, \dots)$ on both sides, one obtains a whole class of expansions for $\prod_{n=1}^{\infty}(1-x^n)^{-1}$ involving the two parameters $k$ and $h$. In particular, taking $k=0$ and carrying out some routine calculations, we get (3.4).

REMARKS. The above technique for obtaining expansions for $\prod_{n=1}^{\infty}(1-x^n)^{-1}$ corresponding to known expansions for $\prod_{n=1}^{\infty}(1+bx^n)$ is, of course, not new. For example, if instead of (3.1) we use Euler's formula ([3, p. 49])

$$\prod_{n=1}^{\infty}(1+bx^{n-1/2}) = \sum_{r=0}^{\infty}\frac{b^r x^{r^2/2}}{(1-x)\dots(1-x^r)} \tag{3.8}$$

to substitute for the products on the right side of (3.7), and then compare the cofficients of $b^k$ on the two sides, we obtain the following sequence of identities of Rademacher ([10, pp. 61–62]). For $k=0, 1, 2, \dots$,

$$\prod_{n=1}^{\infty}(1-x^n)^{-1} = \frac{1}{(1-x)\dots(1-x^k)}+\frac{x^{k+1}}{(1-x)^2(1-x^2)\dots(1-x^{k+1})} + \dots + \frac{x^{l(k+l)}}{(1-x)^2\dots(1-x^l)^2(1-x^{l+1})\dots(1-x^{k+l})}+\dots \tag{3.9}$$

In particular, for $k=0$, we get the identity—due to Euler ([7, pp. 280–281])—

$$\prod_{n=1}^{\infty}(1-x^n)^{-1} = \sum_{n=0}^{\infty} \frac{x^{n^2}}{(1-x)^2(1-x^2)^2\ldots(1-x^n)^2}. \tag{3.10}$$

In their celebrated paper ([6, p. 279]), Hardy and Ramanujan referred to the identity (3.10) and made the cryptic remark that it is "capable of wide generalization—and on elementary algebraic reasoning." Commenting on this, Rademacher ([10, pp. 61–62]) says: "this remark was at first not very obvious to me; but it can now be interpreted in the following way . . .". He then proves (3.9) and says ". . . and we get the 'wide generalization' of which Hardy and Ramanujan spoke". (Further extensions of this identity may be found in [1].) We wish to point out that (3.9) itself is the special case $a=0$, $b=x$ of the (probably not too well known) Cauchy identity ([4, p. 48]):

$$\prod_{n=0}^{\infty}\left(\frac{1-ax^n}{1-bx^n}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n(a-b)(a-bx)\ldots(a-bx^{n-1})}{(1-x)\ldots(1-x^n)\cdot(1-b)\ldots(1-bx^{n-1})}\, x^{n(n-1)/2} \tag{3.11}$$

(Professor L. Carlitz kindly drew our attention to this identity.)

A combinatorial proof of (3.10) is known ([7, p. 281]) and such a proof can be given for (3.9) also. It would be interesting to know if a combinatorial proof can be given for (3.11) also.

## 4. A generalization of (1.1) and (1.5).

In this section we give two brief proofs of the following identity:

$$\sum_{n=0}^{\infty} \frac{(1-\beta)(1-\beta x)\ldots(1-\beta x^{n-1})x^n}{(1-\gamma)(1-\gamma x)\ldots(1-\gamma x^{n-1})} = \left(\beta-\frac{\gamma}{x}\right)^{-1}\left\{1-\frac{\gamma}{x}-\prod_{n=0}^{\infty}\frac{(1-\beta x^n)}{(1-\gamma x^n)}\right\}. \tag{4.1}$$

We note that if $\gamma=0$ we have a slightly altered form of (1.1), and if $\beta=0$ we have a result equivalent to (1.5).

**First proof.** If we set $\alpha=\tau=x$ in [2, p. 576, eq. (I1)], we have

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n \prod_{j=0}^{n-1}\frac{(1-\beta x^j)}{(1-\gamma x^j)} &= \prod_{m=0}^{\infty}\frac{(1-\beta x^m)}{(1-\gamma x^m)} \sum_{n=0}^{\infty}\frac{\beta^n}{1-x^{n+1}}\prod_{j=0}^{n-1}\frac{(1-(\gamma/\beta)x^j)}{(1-x^{j+1})} \\
&= \left(\beta-\frac{\gamma}{x}\right)^{-1}\prod_{m=0}^{\infty}\frac{(1-\beta x^m)}{(1-\gamma x^m)}\left(\sum_{n=0}^{\infty}\beta^n\prod_{j=0}^{n-1}\frac{(1-(\gamma/\beta)x^{j-1})}{(1-x^j)}-1\right) \\
&= \left(\beta-\frac{\gamma}{x}\right)^{-1}\prod_{m=0}^{\infty}\frac{(1-\beta x^m)}{(1-\gamma x^m)}\left(\prod_{n=0}^{\infty}\frac{(1-\gamma x^{n-1})}{(1-\beta x^n)}-1\right) \\
&= \left(\beta-\frac{\gamma}{x}\right)^{-1}\left\{1-\frac{\gamma}{x}-\prod_{m=0}^{\infty}\frac{(1-\beta x^m)}{(1-\gamma x^m)}\right\},
\end{aligned}$$

where the penultimate equation follows from the summation of the ${}_1\phi_0$ [9, p. 92, eq. (3.2,2.12)].

**Second proof.** In (4.1), we replace $x$ by $x^2$, then $\beta$ by $-\beta x$ and $\gamma$ by $\gamma x^2$. Thus we have

$$\sum_{n=0}^{\infty}\frac{(1+\beta x)(1+\beta x^3)\dots(1+\beta x^{2n-1})x^{2n}}{(1-\gamma x^2)(1-\gamma x^4)\dots(1-\gamma x^{2n})} = (\gamma+\beta x)^{-1}\left\{\prod_{m=0}^{\infty}\frac{(1+\beta x^{2m+1})}{(1-\gamma x^{2m+2})}+\gamma-1\right\}. \tag{4.2}$$

Now clearly the coefficient of $x^N\beta^M\gamma^R$ in

$$\prod_{m=0}^{\infty}\frac{(1+\beta x^{2m+1})}{(1-\gamma x^{2m+2})} \tag{4.3}$$

is the number of partitions of $N$ in which there are $M$ odd parts and $R$ even parts with the proviso that no odd parts are repeated. In the same manner

$$\frac{(1+\beta x)(1+\beta x^3)\dots(1+\beta x^{2n-1})\gamma x^{2n}}{(1-\gamma x^2)(1-\gamma x^4)\dots(1-\gamma x^{2n})} \tag{4.4}$$

is the generating function for partitions of the above type when the largest part is $2n$, and

$$\frac{(1+\beta x)(1+\beta x^3)\dots(1+\beta x^{2n-1})\beta x^{2n+1}}{(1-\gamma x^2)(1-\gamma x^4)\dots(1-\gamma x^{2n})} \tag{4.5}$$

when the largest part is $2n+1$. Summing (4.4) and (4.5) over all possible values of $n$ we obtain a new expression for (4.3). Thus

$$1+\sum_{n=1}^{\infty}\frac{(1+\beta x)(1+\beta x^3)\dots(1+\beta x^{2n-1})\gamma x^{2n}}{(1-\gamma x^2)(1-\gamma x^4)\dots(1-\gamma x^{2n})} + \sum_{n=0}^{\infty}\frac{(1+\beta x)(1+\beta x^3)\dots(1+\beta x^{2n-1})\beta x^{2n+1}}{(1-\gamma x^2)(1-\gamma x^4)\dots(1-\gamma x^{2n})} = \prod_{m=0}^{\infty}\frac{(1+\beta x^{2m+1})}{(1-\gamma x^{2m+2})}. \tag{4.6}$$

Combining the two sums in (4.6), we obtain

$$(\gamma+\beta x)\sum_{n=1}^{\infty}\frac{(1+\beta x)(1+\beta x^3)\dots(1+\beta x^{2n-1})x^{2n}}{(1-\gamma x^2)(1-\gamma x^4)\dots(1-\gamma x^{2n})} = \prod_{m=0}^{\infty}\frac{(1+\beta x^{2m+1})}{(1-\gamma x^{2m+2})}-1-\beta x. \tag{4.7}$$

Hence dividing both sides of (4.7) by $\gamma+\beta x$ and then adding 1 to each side, we have (4.2).

5. **An application of the identity** (3.4). Let $p(n)$ denote, as usual, the number of unrestricted partitions of $n$, and $q(n)$ the number of partitions of $n$ into distinct odd parts. We generalize these functions as follows. Let

(i) $p_h(n)$ = the number of partitions of $n$ (repeated parts allowed) such that all the even parts are $\geq 2h$;

(ii) $q_h(n)$ = the number of partitions of $n$ into odd parts which are distinct except the largest part which is repeated exactly $h$ times.

It is clear that $p_1(n)=p(n)$ and $q_1(n)=q(n)$. We now prove the following curious result.

THEOREM 3. *For* $n>h^2-h$,

$$p_h(n) \equiv q_h(n-h^2+h) \pmod 2; \tag{5.1}$$

*in particular,*

$$p(n) \equiv q(n) \pmod 2. \tag{5.2}$$

**Proof.** We utilize the identity (3.4) of Theorem 2 and the fact that for any polynomial $g(x)$ with integer coefficients we have for any (positive or negative) integer $a$,

$$(g(x)^{2a} \equiv (g(x^2))^a \pmod 2. \tag{5.3}$$

Thus applying (5.3) to $D(x)$ defined in (3.5), we get

$$D^2 = (D(x))^2 \equiv x^{h(h-1)}/\{(1-x^2)\dots(1-x^{2h-2})\} \pmod 2.$$

We similarly apply (5.3) to

$$\prod_{j=i+1}^{h-1} (x^{-j}-1)^2$$

and

$$\left(\sum_{n=0}^{\infty} x^{h(n+1/2)}C_{k-h,n}(x)\right)^2,$$

and obtain from (3.4) after some simplification,

$$\begin{aligned}\prod_{n=1}^{\infty}(1-x^n)^{-1} &\equiv \frac{x^{h(h-1)}}{(1-x^2)\dots(1-x^{2h-2})}+\sum_{i=1}^{h-1}\frac{1}{(1-x^2)\dots(1-x^{2i})}\\ &\quad+\frac{x^{h^2}}{(1-x^2)\dots(1-x^{2h-2})}\sum_{k=h}^{\infty}\sum_{n=0}^{\infty}x^{2hn}C_{k-h,n}(x^2) \pmod 2.\end{aligned} \tag{5.4}$$

We now change the order of summation of the double sum on the right side of the above equation and note that

$$\sum_{k=h}^{\infty} C_{k-h,n}(x^2) = \sum_{k=0}^{\infty} c_{k,n}(x^2) = \prod_{i=1}^{n}(1+x^{2i-1}),$$

where in deriving the last equation we use (3.6). Hence (5.4) gives

$$\begin{aligned}\prod_{n=1}^{\infty}(1-x^n)^{-1} &\equiv \frac{x^{h(h-1)}}{(1-x^2)\dots(1-x^{2h-2})}+\sum_{i=1}^{h-1}\frac{1}{(1-x^2)\dots(1-x^{2i})}\\ &\quad+\frac{x^{h^2}}{(1-x^2)\dots(1-x^{2h-2})}\sum_{n=0}^{\infty}x^{2hn}(1+x)(1+x^3)\dots(1+x^{2n-1}),\end{aligned}$$

where the congruences throughout are taken modulo 2.

This gives, in turn,

$$\begin{aligned}\prod_{h=1}^{\infty}{}^{*}(1-x^n)^{-1} &\equiv x^{h(h-1)}+\sum_{i=1}^{h-1}(1-x^{2i+2})\dots(1-x^{2h-2})\\ &\quad+x^{h^2-h}J_h(x)\end{aligned} \tag{5.5}$$

where

$$J_h(x) = \sum_{n=0}^{\infty} x^{h(2n+1)}(1+x)(1+x^3)\dots(1+x^{2n-1}),$$

and $\prod_{n=1}^{\infty *}$ indicates that the product is taken for all natural numbers $n$ except $n=2, 4, \dots, 2h-2$. It is clear that

$$\prod_{n=1}^{\infty *} (1-x^n)^{-1} = 1+\sum_{n=1}^{\infty} p_h(n)x^n,$$

and

$$J_h(x) = 1+\sum_{n=1}^{\infty} q_h(n)x^n.$$

Hence on comparing the coefficients of $x^n$ for $n>h^2-h$ on both sides of (5.5), we obtain the result of Theorem 3.

The fact that $p(n)\equiv q(n) \pmod 2$ is, of course, directly derivable from the observations that $p(n)-q(n)$ enumerates the nonself-conjugate partitions of $n$ [7, p. 279, Theorem 347].

Remarks. A famous unsolved problem in partitions is to characterize all integers $n$ for which $p(n)$ is even. Our result (5.2) shows that this is equivalent to the analogous problem for $q(n)$. It is known that $p(n)$ takes even values and odd values, each for infinitely many $n$. From (5.2) we see that the same property holds for $q(n)$.

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