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MATHEMATICS

AN ARITHMETIC FUNCTION AND AN ASSOCIATED PROBABILITY THEOREM

BY

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1. *The Theorem*: Throughout this note, k and q denote natural numbers satisfying $k>1$ and $0<q<k$. We will prove the

Theorem: The probability that the greatest common divisor of two natural numbers has its k^{th} power free part q^{th} power free is $\zeta(2k)/\zeta(2q)$, where $\zeta(s)$ is Riemann's Zeta function.

Note that the k^{th} power free part of a natural number n becomes n itself when k is infinity, and that when $q=1$ the statement that n is q^{th} power free implies $n=1$.

Corollary 1: The probability that two integers be relatively prime is $6/\pi^2$. This is a well known result (see, for instance [1, theorem 332] and follows when $k=\infty$ and $q=1$.

Corollary 2: The probability that the greatest common divisor of two natural numbers be q^{th} power free is $1/\zeta(2q)$ (McCarthy [2, theorem 4]). This follows on taking $k=\infty$.

Corollary 3: The probability that the greatest common divisor of two natural numbers is a k^{th} power integer is $6\zeta(2k)/\pi^2$. In particular, the probability that the greatest common divisor is a perfect square is $\pi^2/15$. We have only to take $q=1$ in the theorem.

2. *The Function*: $\varphi_{k,q}(n)$. Let $\varphi_{k,q}(n)$ denote the number of integers $m(\text{mod}\, n)$ such that (m, n) has its k^{th} power free part q^{th} power free, with $\varphi_{k,q}(1)=1$.

Let $n>1$ have the canonical form

$$n=p_1^{a_1}\, p_2^{a_2} \dots p_r^{a_r}. \tag{2.1}$$

It is not difficult to see that

$$\varphi_{k,q}(n)=n \prod_{p_i}\left(1-\frac{1}{p_i^{q}}+\frac{1}{p_i^{k}}-\frac{1}{p_i^{k+q}}+\frac{1}{p_i^{2k}}-\dots\right)$$

where the expansions terminate with the terms whose denominators are the largest factors of n of the forms $p_i^{t_i k}$ or $p^{t_i k+q}$. Let $\lambda_{k,q}(n)$ be the multiplicative function defined for prime powers p^a as follows:

$$\lambda_{k,q}(p^a) = \begin{cases} 1, & a \equiv 0 \pmod{k} \\ -1, & a \equiv q \pmod{k} \\ 0 & \text{otherwise.} \end{cases}$$

We easily observe that

$$\varphi_{k,q}(n) = \sum_{d|n} \lambda_{k,q}(d)\ (n/d). \tag{2.2}$$

Since one can verify that

$$\sum \frac{\lambda_{k,q}(n)}{n^s} = \zeta(ks)/\zeta(qs), \tag{2.3}$$

we have

$$\sum \frac{\varphi_{k,q}(n)}{n^s} = \zeta(s-1)\ \zeta(ks)/\zeta(qs). \tag{2.4}$$

It is also of interest to note the following other defining properties of $\varphi_{k,q}(n)$.

$$\varphi_{k,q}(n) = \sum_{\substack{d/n \\ d \epsilon s}} \varphi(n/d) \tag{2.5}$$

where s is the set consisting of unity and all natural numbers $n > 1$ in whose canonical representations given by (2.1) none of the exponents a_i satisfies any of the $k-q$ congruences

$$a_i \equiv t \pmod{k},\ t = q,\ q+1,\ \dots,\ k-1.$$

Also

$$\varphi_{k,q}(n) = \sum_{\substack{r (\text{mod } n) \\ (r,\, n)\, \epsilon\, s}} 1. \tag{2.6}$$

Finally we note without proof that $\varphi_{k,q}(n)$

may also be defined as the number of those arithmetic progressions $r \pmod{n}$ $(r = 1, 2, \dots, n)$ which contain an infinity of terms whose greatest common divisors with n have their k^{th} power free parts q^{th} power free.

The functions $\varphi_{k,q}(n)$ and $\lambda_{k,q}(n)$ were considered in a recent paper of V. C. Harris and the author [3].

3. *Proof of the Theorem*: We follow the method of Hardy and Wright and McCarthy. Since the number of pairs of natural numbers (a, b) for which $b > 0$ and $0 < a \leqslant b < n$ is $n(n+1)/2$, and the number of such pairs

for which the greatest common divisor (a, b) has its k^{th} power free part q^{th} power free is

$$\sum_{r=1}^{n} \varphi_{k,q}(r),$$

the theorem follows from the

Lemma:

$$\sum_{r=1}^{n} \varphi_{k,q}(r) = \tfrac{1}{2} n^2 \zeta\,(2k)/\zeta\,(2q) + 0\,(n \log n).$$

To show this, we make use of the relation (2.2) and familiar arguments. We have

$$\begin{aligned}
\sum_{=1}^{n} \varphi_{k,q}(r) &= \sum_{m=1}^{n} \sum_{d\,d'=m} \lambda(d)d' \\
&= \sum_{d\,d' \leqslant n} \lambda(d)d' = \sum_{d=1}^{n} \lambda(d) \sum_{d'=1}^{[n/d]} d' \\
&= 1/2 \sum_{d=1}^{n} \lambda(d)[n/d]([n/d]+1) \\
&= n^2/2 \sum_{d=1}^{\infty} \lambda(d)/d^2 - n^2/2 \sum_{d=n+1}^{\infty} \lambda(d)/d^2 \\
&\quad + 0(n \sum_{d=1}^{n} 1/d) \\
&= 1/2 n^2 \zeta(2k)/\zeta(2q) + 0(n) + 0(n \log n).
\end{aligned}$$

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