AN ARITHMETIC FUNCTION AND AN ASSOCIATED PROBABILITY THEOREM

BY

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1. The Theorem: Throughout this note, \( k \) and \( q \) denote natural numbers satisfying \( k > 1 \) and \( 0 < q < k \). We will prove the

**Theorem:** The probability that the greatest common divisor of two natural numbers has its \( k \)th power free part \( q \)th power free is \( \zeta(2k)/\zeta(2q) \), where \( \zeta(s) \) is Riemann's Zeta function.

Note that the \( k \)th power free part of a natural number \( n \) becomes \( n \) itself when \( k \) is infinity, and that when \( q = 1 \) the statement that \( n \) is \( q \)th power free implies \( n = 1 \).

**Corollary 1:** The probability that two integers be relatively prime is \( 6/\pi^2 \). This is a well known result (see, for instance [1, theorem 332]) and follows when \( k = \infty \) and \( q = 1 \).

**Corollary 2:** The probability that the greatest common divisor of two natural numbers be \( q \)th power free is \( 1/\zeta(2q) \) (McCarthy [2, theorem 4]). This follows on taking \( k = \infty \).

**Corollary 3:** The probability that the greatest common divisor of two natural numbers is a \( k \)th power integer is \( 6\zeta(2k)/\pi^2 \). In particular, the probability that the greatest common divisor is a perfect square is \( \pi^2/15 \). We have only to take \( q = 1 \) in the theorem.

2. The Function: \( \varphi_{k,q}(n) \). Let \( \varphi_{k,q}(n) \) denote the number of integers \( m \mod n \) such that \( (m, n) \) has its \( k \)th power free part \( q \)th power free, with \( \varphi_{k,q}(1) = 1 \).

Let \( n > 1 \) have the canonical form

\[
n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}.
\]

It is not difficult to see that

\[
\varphi_{k,q}(n) = \prod_{p_i} \left( 1 - \frac{1}{p_i^q} + \frac{1}{p_i^{k+q}} - \frac{1}{p_i^{2k+q}} + \ldots \right)
\]

where the expansions terminate with the terms whose denominators are the largest factors of \( n \) of the forms \( p_i^{i+k} \) or \( p_i^{i+k+q} \). Let \( \lambda_{k,q}(n) \) be the
multiplicative function defined for prime powers $p^a$ as follows:

$$
\lambda_{k, g}(p^a) = \begin{cases} 
1, & a \equiv 0 \pmod{k} \\
-1, & a \equiv g \pmod{k} \\
0, & \text{otherwise.}
\end{cases}
$$

We easily observe that

$$(2.2) \quad \varphi_{k, g}(n) = \sum_{d|n} \lambda_{k, g}(d) \frac{n}{d}.$$ 

Since one can verify that

$$(2.3) \quad \sum \frac{\lambda_{k, g}(n)}{n^s} = \zeta(ks)/\zeta(qs),$$

we have

$$(2.4) \quad \sum \frac{\varphi_{k, g}(n)}{n^s} = \zeta(s-1) \zeta(ks)/\zeta(qs).$$

It is also of interest to note the following other defining properties of $\varphi_{k, g}(n)$.

$$(2.5) \quad \varphi_{k, g}(n) = \sum_{d|n} \varphi(n/d)$$

where $s$ is the set consisting of unity and all natural numbers $n > 1$ in whose canonical representations given by (2.1) none of the exponents $a_i$ satisfies any of the $k-q$ congruences

$$a_i \equiv t \pmod{k}, \quad t = q, q+1, \ldots, k-1.$$ 

Also

$$(2.6) \quad \varphi_{k, g}(n) = \sum_{r \equiv a \pmod{n}} 1.$$ 

Finally we note without proof that $\varphi_{k, g}(n)$ may also be defined as the number of those arithmetic progressions $r \pmod{n}$ ($r = 1, 2, \ldots, n$) which contain an infinity of terms whose greatest common divisors with $n$ have their $k$th power free parts $q$th power free.

The functions $\varphi_{k, g}(n)$ and $\lambda_{k, g}(n)$ were considered in a recent paper of V. C. Harris and the author [3].

3. Proof of the Theorem: We follow the method of Hardy and Wright and McCarthy. Since the number of pairs of natural numbers $(a, b)$ for which $b > 0$ and $0 < a < b < n$ is $n(n+1)/2$, and the number of such pairs
for which the greatest common divisor \((a, b)\) has its \(k\)th power free part
\(q\)th power free is
\[
\sum_{r=1}^{n} \varphi_{k,q}(r),
\]
the theorem follows from the
Lemma:
\[
\sum_{r=1}^{n} \varphi_{k,q}(r) = \frac{1}{2} n^2 \zeta(2k) / \zeta(2q) + 0 (n \log n).
\]
To show this, we make use of the relation (2.2) and familiar arguments. We have
\[
\sum_{r=1}^{n} \varphi_{k,q}(r) = \sum_{m=1}^{n} \sum_{d \leq m} \lambda(d)d'
\]
\[
= \sum_{d \leq m} \lambda(d)d' = \sum_{d=1}^{n} \lambda(d) \sum_{d' \leq \frac{n}{d}} d'
\]
\[
= 1/2 \sum_{d=1}^{n} \lambda(d) \left[ \frac{n}{d} \right] \left( \left[ \frac{n}{d} \right] + 1 \right)
\]
\[
= n^2/2 \sum_{d=1}^{\infty} \frac{\lambda(d)}{d^2} - n^2/2 \sum_{d=n+1}^{\infty} \frac{\lambda(d)}{d^2}
\]
\[
+ 0(n \sum_{d=1}^{n} 1/d)
\]
\[
= 1/2 n^2 \zeta(2k) / \zeta(2q) + 0(n) + 0(n \log n).
\]

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REFERENCES