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ON A GENERALIZED WARING'S PROBLEM IN ALGEBRAIC
NUMBER FIELDS

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1. Introduction.

Let $K$ be a totally real agebraic number field of degree $n$. Let $K^{(i)} (1 \leq i \leq n)$ be the
conjugate fields of $K$. For $\gamma \in K$, we denote by $\gamma^{(i)} (1 \leq i \leq n)$ the conjugates of $\gamma$ and
$N(\gamma) = \prod_{i=1}^{n} \gamma^{(i)}$ the norm of $\gamma$. Let $\gamma_j (1 \leq j \leq n)$ be numbers of $K$ and $x_i (1 \leq i \leq n)$ be
real numbers. We set $\xi = \sum_{j=1}^{n} x_j \gamma_j$ and define $\xi^{(i)} = \sum_{j=1}^{n} x_j \gamma_j^{(i)} (1 \leq i \leq n)$. We use the
notations

$$||\xi|| = \max_i |\xi^{(i)}|, \quad S(\xi) = \sum_{i=1}^{n} \xi^{(i)} \quad \text{and} \quad E(\xi) = \exp(2\pi i S(\xi)).$$

where $\exp(x) = e^x$. A number $\gamma$ of $K$ is called totally nonnegative if $\gamma^{(i)} \geq 0 (1 \leq i \leq n)$.

It was Siegel [5,6] who succeeded in dealing with Waring's problem in an arbitrary
algebraic number field by his generalized circle method, and obtained the result corre-
sponding to Hardy-Littlewood's estimation on $G(k)$.

Ayoub [1] gave an extension of Siegel's theorem, namely to replace the $k$th powers by
polynomial summands for totally real algebraic number fields. Let $\nu$, $\alpha$, $\alpha_i (1 \leq i \leq k-1)$
be nonzero totally nonnegative integers of $K$ and $k \geq 2$. Consider the polynomial

$$\phi(\xi) = \alpha \xi^k + \alpha_{k-1} \xi^{k-1} + \cdots + \alpha_1 \xi$$

and the equation

(1) \hspace{1cm} \nu = \phi(\xi_1) + \cdots + \phi(\xi_n).

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Canada.
Let $A(\nu)$ be the number of solutions of (1) in totally nonnegative integers $\xi_1, \ldots, \xi_s$ satisfying $N(\xi_i) \leq N(\nu)^{-1/k} = 1 (1 \leq i \leq s)$. Ayoub established that if $s \geq n(2^k + n) + 1$, then

$$A(\nu) = D^{1/2(1-s)} \mathcal{S}(\nu) \left( \frac{\Gamma(1 + \frac{1}{k})}{\Gamma(s/k)} \right)^n N(\alpha)^{-s/k} N(\nu)^{-1+s/k} (1 + o(1))$$

where $D$ is the absolute value of the discriminant of $K$ and $\mathcal{S}(\nu)$ the singular series of (1).

It is our object to give a natural extension of Ayoub's theorem, namely to replace the $\phi(\xi)$ by different polynomials of degree $k$, and to give a slight improvement on the lower bound for $s$ to $\max(4kn, 2^k + 1)$. In addition, it seems that there is a gap in his proof of (2) (See Remark in §6).

Consider the polynomials

$$\phi_i(\lambda) = \alpha_{ki} \lambda^{k-1} + \alpha_{k-1,i} \lambda^{k-1} + \cdots + \alpha_i \lambda, \quad 1 \leq i \leq s$$

and the equation

$$\nu = \phi_1(\lambda_1) + \cdots + \phi_s(\lambda_s),$$

where $\nu$ and $\alpha_{ki} (1 \leq i \leq s)$ are given nonzero totally nonnegative integers and $\alpha_{ji} (1 \leq j \leq k - 1, 1 \leq i \leq s)$ are integers. Let $B(\nu)$ be the number of solutions of (4) in totally nonnegative integers $\lambda_i (1 \leq i \leq s)$ satisfying $N(\lambda_i) \leq N(\nu)^{-1/k} = 1 (1 \leq i \leq s)$.

**Theorem.** If $s \geq \max(4kn, 2^k + 1)$, then

$$B(\nu) = \mathcal{S}' J N(\nu)^{-1+s/k} (1 + o(1))$$

where $\mathcal{S}'$ and $J$ denote the singular series and singular integral of (4) respectively. (See §5).

2. The generalized circle method.

Let $\omega_1, \ldots, \omega_n$ be an integral basis of $K$ and $\delta$ the different of $K$. We can choose a basis $\rho_1, \ldots, \rho_n$ of $\delta^{-1}$ such that
\[ S(\rho_i, \omega_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \]

Set \( \xi = x, \rho_1 + \cdots + x_n \rho_n \) and \( \eta = y_1 \omega_1 + \cdots + y_n \omega_n \) where \( x_i \) and \( y_i \) \( (1 \leq i \leq n) \) are real numbers. We denote by \( dx = \prod_{i=1}^{n} dx_i \), \( dy = \prod_{i=1}^{n} dy_i \), \( P(T) \) the set of \( y = (y_1, \ldots, y_n) \) satisfying \( 0 \leq \eta^{(i)} \leq T \) \((1 \leq i \leq n)\) and \( \sum_{\lambda \in \Lambda(T)} \) a sum where \( \lambda \) runs over all integers such that \( 0 \leq \lambda^{(i)} \leq T \) \((1 \leq i \leq n)\).

Let \( Q \) denote the rational number field and \( U_n \) the \( n \)-dimensional unit cube \( \{ \bar{x} = (x_1, \ldots, x_n) : 0 \leq x_i < 1 (1 \leq i \leq n) \} \). Let \( h \) and \( t \) be real numbers satisfying \( h > 2Dt \) and \( t > 1 \). For any \( \gamma \in K \), we can determine uniquely two integral ideals \( A_0, L \) such that

\[ \gamma^6 = L/A, \ (A, L) = 1. \]

We write \( \gamma \rightarrow A \). Let \( \Gamma(t) \) be the set consisting of \( \gamma = x_1 \rho_1 + \cdots + x_n \rho_n \) satisfying

\[ \bar{x} \in U_n : x_i \in Q(1 \leq i \leq n), \ \gamma \rightarrow A \ \text{and} \ N(A) \leq t^n. \]

For every \( \gamma \in \Gamma(t) \) subject to \( \gamma \rightarrow A \), we define the basis domain \( B_{\gamma} \) by

\[ \bar{x} : \bar{x} \in U_n, \ \prod_{i=1}^{n} \max \{ h | \xi^{(i)} - \gamma^{(i)} |, t^{-1} \} \leq N(A)^{-1} \ \text{for some } \gamma_0 \equiv \gamma (\mod \delta^{-1}). \]

We can show that if \( \gamma_1 \) and \( \gamma_2 \) belong to \( \Gamma(t) \) and \( \gamma_1 \neq \gamma_2 \), then

\[ B_{\gamma_1} \cap B_{\gamma_2} = \phi. \]

(See, Siegel [5,6] or Wang [9]). We set

\[ B = \bigcup_{\gamma \in \Gamma(t)} B_{\gamma} \]

and define the supplementary domain \( S \) of \( B \) with respect to \( U_n \) by

\[ S = U_n - B. \]

This division of \( U_n \) into \( B \) and \( S \) depends on \( (h, t) \). We shall call this division the Farey division of \( U_n \) with respect is \( (h, t) \).
Let
\[ S_i(\xi, T) = S_i(\xi) = \sum_{\lambda \in \mathcal{P}(T)} E(\phi_i(\gamma)\xi) \text{ and } S(\xi, T) = \prod_{i=1}^{s} S_i(\xi). \]

Since for any integer \( \alpha \)
\[ \int_{U_n} E(\alpha \xi) dx = \begin{cases} 1, & \text{if } \alpha = 0, \\ 0, & \text{if } \alpha \neq 0, \end{cases} \]
the number of solutions of (4) in totally nonnegative integers \( \lambda_i(1 \leq i \leq s) \) satisfying \( \|\lambda_i\| \leq T(1 \leq i \leq s) \) is equal to
\[ Z(\nu) = \int_{U_n} S(\xi, T) E(-\nu \xi) dx = \int_{B} S(\xi, T) E(-\nu \xi) dx + \int_{S} S(\xi, T) E(-\nu \xi) dx \]
\[ = Z_1 + Z_2, \]
say.

3. Asymptotic expansion for \( S_i(\xi) \).

Denote by
\[ a = \frac{1}{4n} + \frac{1}{4kn}, \quad t = T^{1-s} \quad \text{and} \quad h = T^{k-1+a} \]
where \( T > 1 \). For any \( \gamma \in \Gamma(i) \), let \( \mathcal{L}_\ell = (\alpha_{\ell}\gamma, \ldots, \alpha_{\ell}\gamma) \) be the ideal generated by \( \alpha_{\ell}\gamma, \ldots, \alpha_{\ell}\gamma \) and \( \mathcal{A}_\ell \) denote the denominator of \( \mathcal{L}_\ell \). We use the notations
\[ \xi - \gamma = \xi, \quad G_\xi(\gamma) = N(\mathcal{A}_\ell)^{-1} \sum_{\lambda(\mathcal{L}_\ell)} E(\phi_\lambda(\gamma)\xi) \text{ and } I_\ell(\gamma; T) = \int_{\mathcal{P}(T)} E(\phi_\lambda(\eta)\xi) d\eta, \]
where \( \sum_{\lambda(\mathcal{L}_\ell)} \) denotes a sum in which \( \lambda \) runs over a complete residue system mod \( \mathcal{A}_\ell \). We use \( E_n \) to denote the whole \( n \)-dimensional Euclidean space.

**Lemma 1.** (Hua [3]). For any given \( \epsilon > 0 \),
\[ G_\ell(\gamma) \ll N(\mathcal{A}_\ell)^{\epsilon-1/2}, \]
hereafter the constants implicit in \( \ll \) or \( \ll \) may depend on \( \epsilon \) and \( K \). (Note that \( \alpha_{\ell}\)'s are given integers of \( K \)).

**Lemma 2.** (Vinogradov [8]). Let \( f(x) = \gamma_kx^k + \cdots + \gamma_1x \) be a polynomial with real coefficients. Then
\[
\int_0^1 \exp(2\pi i f(x)) dx \ll \min(1, |\gamma_1|^{-1/k}, \ldots, |\gamma_k|^{-1/k}).
\]

**Lemma 3.** \(I_\ell(\zeta, T) \ll \prod_{i=1}^n \min(T, |\zeta|^{-1/k}).\)

**Proof:** Let \(\eta^{(j)} = T u_j, 1 \leq j \leq n.\) Then the Jacobian of \(y_1, \ldots, y_n\) with respect to \(u_1, \ldots, u_n\) is equal to \(D^{-1/2} T^n.\) Therefore by Lemma 2,

\[
I_\ell(\zeta, T) = D^{-1/2} T^n \prod_{j=1}^n \int_0^1 \exp(2\pi i \nu_j(u_j T) \zeta^{(j)}) du_j \ll \prod_{j=1}^n \min(T, |\zeta|^{-1/k}),
\]

and the lemma is proved.

**Lemma 4.** If \(s < k,\) then

\[
\int_{E_n} \prod_{i=1}^n \min(T, |\zeta|^{-1/k})^s dx \ll T^{(s-k)n}.
\]

**Proof:** Let \(\zeta^{(j)} = v_j, 1 \leq j \leq n.\) Then the Jacobian of \(x_1, \ldots, x_n\) with respect to \(v_1, \ldots, v_n\) is \(D^{1/2}.\) Therefore

\[
\int_{E_n} \prod_{j=1}^n \min(T, |\zeta^{(j)}|^{-1/k})^s dx \ll D^{1/2} \prod_{j=1}^n \int_0^\infty \min(T, v_j^{-s/k}) dv_j
\]

\[
\ll \left( \int_0^{T^{-k}} T^s dv + \int_{T^{-k}}^\infty v^{-s/k} dv \right)^n \ll T^{(s-k)n}
\]

and the lemma is proved.

**Lemma 5.** Let \(\mu\) be an integer. Then

\[
\sum_{\gamma+\mu \in E(T)} E(\nu_0(\lambda + \mu) \zeta) + N(A\ell)^{-1} I_\ell(\zeta, T) + O(N(A\ell)^{-1} T^{m-s}).
\]

**Proof:** The proof is similar to the proof for the sum of \(E(\alpha(\lambda + \mu) \zeta)\) (See, Siegel [5, 6] or Wang [9]). We give the proof here, for the sake of completeness.

Determine positive numbers \(a^{(i)} (1 \leq i \leq n)\) such that

\[
(6) \quad a^{(i)} \max(h_i^{(i)}(\zeta, T)) = D^{1/2n} \prod_{i=1}^n \max(h_i^{(i)}(\zeta, T^{-1/n})^{1/n} N(A\ell)^{1/n}
\]

then
\[ \prod_{i=1}^{n} \varrho^{(i)} = D^{1/2} N(A_\ell), \]

and it follows by Minkowski's linear form theorem that there exists \( \sigma \in A_\ell \) such that

\[ 0 < |\sigma^{(i)}| \leq \varrho^{(i)}, \quad 1 \leq i \leq n. \]

Hence \( \sigma A_\ell^{-1} = \mathcal{L}_\ell \) is an integral ideal and

\[ N(\mathcal{L}_\ell) = |N(\sigma)N(A_\ell)^{-1}| \leq \prod_{i=1}^{n} \varrho^{(i)} N(A_\ell)^{-1} = \sqrt{D}. \]

Therefore \( \mathcal{L}_\ell \) belongs to a finite set depending on \( K \) only. Let \( \sigma_1, \ldots, \sigma_n \) be a basis of \( \mathcal{L}_\ell^{-1} \). Then \( \mathcal{L}_\ell = \sigma \mathcal{L}_\ell^{-1} \) has a basis

\[ \tau_i = \sigma \sigma_i, \quad 1 \leq i \leq n. \]

By (6), (7) we obtain

\[ \| \tau_i \| = O(\| \sigma \|) = O(\theta) = O(t). \]

Expressing \( \lambda \) in terms of \( \tau_i \)'s we have

\[ \lambda = g_1 \tau_1 + \cdots + g_n \tau_n, \]

where \( g_i \)'s are rational integers. Let \( U(\lambda) \) denote the unit cube

\[ \varepsilon : \tau = s_1 \tau_1 + \cdots + s_n \tau_n, \quad g_i \leq s_i < g_i + 1 (1 \leq i \leq n) \]

and \( ds = \prod_{i=1}^{n} ds_i \). Then by (6) we have

\[ E(\varphi_\ell(\lambda + \mu) \varsigma) - \int_{U(\lambda)} E(\varphi_\ell(r + \mu) \varsigma) ds \leq \| (\lambda - r) \varsigma \| (\| r + \mu \|^{-k-1} + \| \lambda + \mu \|^{-k-1} + 1) \]

\[ \leq \| \theta \| T^{-k-1} \leq h^{-1} T^{-k-1} \leq T^{-\alpha}. \]

Since the number of integers \( \lambda \) with \( A_\ell | \lambda \) and \( \lambda + \mu \in P(T) \) is \( O(N(A_\ell)^{-1} T^n) \), we have

\[ \sum_{\lambda + \mu \in P(T) \atop A_\ell | \lambda} E(\varphi_\ell(\lambda + \mu) \varsigma) = \sum_{\lambda + \mu \in P(T) \atop A_\ell | \lambda} \int_{U(\lambda)} E(\varphi_\ell(r + \mu) \varsigma) ds + O(N(A_\ell)^{-1} T^{n-\alpha}) \]
\[
= \int_{0 \leq \theta + r(t) \leq T} \ldots \int E(\varphi_\ell(t + \mu)\zeta)\,ds + O(N(A_\ell)^{-1}T^{n-\alpha}).
\]

Let \( r + \mu = \eta \). Since the Jacobian of \( \alpha_i \)'s with respect to \( y_i \)'s is \( D^{1/2} |\det(y_i^{(1)})|^{-1} = N(A_\ell)^{-1} \), the lemma follows.

**Lemma 6.** \( S_\ell(\xi) = G_\ell(\gamma)I_\ell(\xi, T) + O(T^{n-\alpha}). \)

**Proof:** By Lemma 5,

\[
S_\ell(\xi) = \sum_{\mu(A_\ell)} E(\psi_\ell(\mu)\gamma) \sum_{\lambda+\mu \in \mathbb{P}(T)} E(\psi(\lambda + \mu)\zeta)
= \sum_{\mu(A_\ell)} E(\psi_\ell(\mu)\gamma)(N(A_\ell)^{-1}I_\ell(\xi, T) + O(N(A_\ell)^{-1}T^{n-\alpha}))
= G_\ell(\gamma)I_\ell(\xi, T) + O(T^{n-\alpha}).
\]

The lemma is proved.

4. Basic domains.

Set \( G(\gamma) = \prod_{1} G_i(\gamma), \quad S(\xi, T) = \prod_{1} S_i(\xi) \) and \( I(\xi, T) = \prod_{1} I_i(\xi, T) \).

**Lemma 7.** If \( s \geq 4kn \), then

\[
\int_{B} S(\xi, T)E(-\nu\xi)\,dx = \sum_{\gamma \in \Gamma(0)} G(\gamma)E(-\nu\gamma)\int_{B_i} I(\xi, T)E(-\nu\xi)\,dx + O(T^{(s-k)n-\alpha}).
\]

**Proof:** Let \( \gamma \rightarrow A \). Since \( A_i|A|a_{k1} \ldots a_{k1}A_i \), we have

\[
N(A_i)^{-1} \ll N(A)^{-1} \ll N(A_i)^{-1}, \quad 1 \leq i \leq s.
\]

By Lemma 1, 3 and 6, we have

\[
S(\xi, T) = G(\gamma)I(\xi, T) + O\left( T^{n-\alpha}\max_{i} (T^{n-\alpha}, |G_j(\gamma)I_j(\xi, T)|)^{s-1} \right)
= G(\gamma)I(\xi, T) + O(T^{n-\alpha}) + O(T^{n-\alpha}(N(A)^{1/k+s/\alpha}) \prod_{i=1}^{N} \min(T, |\xi|^{-1/k})^{s-1}).
\]

Since the number of \( \gamma \in \Gamma(1) \) such that \( \gamma \rightarrow A \) is \( O(N(A)) \) and the number \( A \) such that \( N(A) = m \) is \( O(m^s) \), we have
(8) \[ \sum_{\gamma \in \Gamma(4)} N(\gamma)^{-\frac{k-1}{a}} \ll \sum_{m \leq \gamma} m^{-\frac{4k-1}{k} + 1/2a} \ll 1, \]

and by Lemma 4 and \( a = \frac{1}{k} + \frac{1}{4k-a} \),

\[ \int_B S(\xi, T)E(-\nu \xi)dx = \sum_{\gamma \in \Gamma(4)} G(\gamma) E(-\nu \gamma) \int_{B_1} I(\xi, T)E(-\nu \xi)dx \]

\[ + O(T^{(a-a)k}) + O(T^{a-k} \sum_{\gamma \in \Gamma(4)} N(\gamma)^{-\frac{k-1}{a}} \int_{E_0} \prod_{i=1}^n \min(T, |z^{(i)|-1/k})^{a-k} dx) \]

\[ = \sum_{\gamma \in \Gamma(4)} G(\gamma) E(-\nu \gamma) \int_{B_1} I(\xi, T)E(-\nu \xi)dx + O(T^{(a-k)\alpha}) \]

The lemma is proved.

**Lemma 8.** If \( s \geq 4kn \), then

\[ \int_B S(\xi, T)E(-\nu \xi)dx = \sum_{\gamma \in \Gamma(4)} G(\gamma) E(-\nu \gamma) \int_{E_0} I(\xi, T)E(-\nu \xi)dx + O(T^{(a-k)a-a}). \]

**Proof:** By Lemma 7, it suffices to show that

\[ W = \sum_{\gamma \in \Gamma(4)} G(\gamma) E(-\nu \gamma) \int_{E_0 - B_1} I(\xi, T)E(-\nu \xi)dx = O(T^{(a-k)a-a}) . \]

If \( \xi \in E_0 - B_1 \), then there is at least one index \( i \) such that \( h|z^{(i)}| > N(\gamma)^{1/a} \). Hence by Lemma 3

\[ \int_{E_0 - B_1} I(\xi, T)E(-\nu \xi)dx \ll \int_{E_0 - B_1} \prod_{i=1}^n \min(T, |z^{(i)}|^{-1/k})dx \]

\[ \ll \int_{h^{-1/2}N(\gamma)^{-1/a}} u^{a/k} du \left( \int_0^{\infty} \min(T, v^{-1/k}) dv \right)^{a-1} \]

\[ \ll h^{a/k-1} N(\gamma)^{a(1-1) + (a-k)(a-1).} \]

By Lemma 1, we have

\[ \sum_{\gamma \in \Gamma(4)} G(\gamma) N(\gamma)^{1/a} \ll \sum_{\gamma \in \Gamma(4)} N(\gamma)^{a-k+1/a} \ll t^{2n}. \]

Therefore

\[ W \ll T^{(a-k)a-1} = t^{2n} . \]
The lemma is proved.

5. Some more lemmas.

Set

\[ J_i(\xi, T) = \int_{\mathcal{P}(T)} E(\alpha \eta^k \xi) dy \quad (1 \leq i \leq s) \quad \text{and} \quad J(\xi, T) = \prod_{i=1}^{s} J_i(\xi, T) . \]

Since

\[ E(\phi(\eta) \xi) - E(\alpha \eta^k \xi) \ll S(|\xi|) T^{k-1} \]

we have for \( \theta = -k + \frac{1}{n+1} \),

\[ \int_{\mathcal{E}_n} (I(\xi, T) - J(\xi, T)) E(-\nu \xi) d\xi \ll \int_{[0] \times \mathcal{T}^n} |I(\xi, T) - J(\xi, T)| d\xi + \sum_{i=1}^{n} \int_{[0] \times \mathcal{T}^n} (|I(\xi, T) - J(\xi, T)| d\nu = \Sigma_1 + \Sigma_2, \text{ say}. \]

By Lemma 3, we have

\[ \sum_{1} \ll \int_{0}^{T^k} \cdots \int_{0}^{T^k} T^{k-1} \sum_{j=1}^{n} v_j \prod_{i=1}^{n} \min(T, v_j^{-1/k})^{s-1} d\nu_1 \cdots d\nu_n \]

\[ \ll T^{(s-1)n+k-1+(n+1)\theta} \ll T^{(s-k)n-n} \]

and for \( s \geq 4kn \),

\[ \Sigma_2 \ll \int_{T^k}^{\infty} T^{-s/k} d\nu \int_{\mathcal{E}_n-1} \prod_{j=1}^{n-1} \min(T, v_j^{-1/k})^{s} d\nu_1 \cdots d\nu_{n-1} \]

\[ \ll T^{-(s-k)(n+1)} \ll T^{(s-k)n-k-1} \ll T^{(s-k)n-1} . \]

By (8) and Lemmas 1 and 8, we have

**LEMMA 9.** If \( s \geq 4kn \), then

\[ \int_{\mathcal{E}_n} S(\xi, T) E(-\nu \xi) d\xi = \sum_{\tau \in \Gamma(t)} G(\gamma) E(\nu \tau) \int_{\mathcal{E}_n} J(\xi, T) E(-\nu \xi) d\xi \]

\[ + o(T^{(s-k)n-n}) . \]

Let \( \eta' = y_1' \omega_1 + \cdots + y_n' \omega_n \), \( \zeta' = z_1' \rho_1 + \cdots + z_n' \rho_n \), \( dy' = dy_1' \cdots dy_n' \), \( dz' = dz_1' \cdots dz_n' \), \( \eta = T \eta' \) and \( \zeta = T^{-k} \zeta' \). The Jacobian of \( y_1, \ldots, y_n \) and \( z_1, \ldots, z_n \) with
respect to \( y'_i, \ldots, y'_n \) and \( x'_1, \ldots, x'_n \) are \( T^n \) and \( T^{-kn} \) respectively. Then
\[
\alpha_k \eta^k \zeta = \alpha_k \eta^k \zeta'.
\]
Write \( \eta' \) and \( \zeta' \) by \( \eta \) and \( \zeta \) again. Then
\[
(9) \quad \int_{\mathbb{R}^n} J(\gamma; T) E(-\nu \zeta) dx = T^{(s-k)n} J(\mu),
\]
where \( \mu = \nu T^{-k} \) and
\[
J(\mu) = \int_{\mathbb{R}^n} \left( \prod_{i=1}^s J_i(\gamma, 1) \right) E(-\mu \zeta) dx.
\]
\( J \) is called the singular integral which is absolutely convergent by Lemma 3 if \( s > k \).

Similar to Siegel [5,6] or Tatzawa [7], Wang [9], we have
\[
J = D^{(\frac{1}{2}(s-k))} k^{-sn} N(\alpha_{k1} \cdots \alpha_{ks})^{1/k} \prod_{\ell=1}^n F_\ell,
\]
where
\[
F_\ell = \int_{W_\ell} \prod_{i=1}^s \omega_i^{\ell-1} dw
\]
in which \( dw = \prod_{i=1}^{s-1} dw_i \) and \( W_\ell \) denotes the domain
\[
0 \leq \omega_i \leq \alpha_{i\ell} (1 \leq i \leq s), \quad \mu(\ell) = \omega_1 - \cdots - \omega_s = 0.
\]
By Lemma 1 we have for \( s \geq 4kn \)
\[
\sum_{\gamma \in \Omega(\ell)} G(\gamma) E(-\nu \gamma) \ll \sum_{\gamma \in \Omega(\ell)} N(\Omega)^{-\frac{1}{2}+\varepsilon} \ll T^{-(s-k)n}.
\]
and thus
\[
(10) \quad \sum_{\gamma \in \Omega(\ell)} G(\gamma) E(-\nu \gamma) = \mathcal{E}' + O(T^{-(s-k)n})
\]
where
\[
\mathcal{E}' = \sum_{\gamma} G(\gamma) E(-\nu \gamma)
\]
in which \( \gamma \) runs over a complete residue system of \((\Omega \delta)^{-1}, \mod \delta^{-1}\), \( \mathcal{E}' \) is called the singular series.

By Lemma 9, (9), (10) and (11), we have

**Lemma 10.** If \( s \geq 4kn \), then
\[ \int_B \delta(\xi, T) E(-\nu \xi) dx = \Theta J T^{(e-k)n} (1 + o(1)). \]

6. Proof of the theorem.

Let \( \theta \) be a number satisfying \( 0 < \theta < (1 - \alpha)2^{1-k} \).

**Lemma 11.** If \( \xi \in S \), then

\[ S_i(\xi) \ll T^{a-\theta} \]

where the constant implicit in \( \ll \) depends on \( \theta \).

This can be proved a a Siegel's lemma [5,6] on the diophantine approximation for \( \xi \in S \) and a theorem of Mitsui [5]. See, Tatzawa [7], p. 54.

**Lemma 12.** If \( 1 \leq j \leq k \), then

\[ \int_{U_0} |S_i(\xi)|^2 dx \ll T^{(e-j)n+\epsilon}. \]

This is a generalization of Hua's inequality [2] in algebraic number fields. See, Ayoub [1], p. 447.

**Lemma 13.** If \( s \geq 2^k + 1 \), then

\[ \int_S S(\xi, T) E(-\nu \xi) dx \ll T^{(e-k)-\theta}. \]

**Proof:** Put

\[ \theta_1 = \frac{\theta}{2} + \frac{1 - \alpha}{2^k}. \]

Then

\[ \theta < \theta_1 < (1 - \alpha)2^{1-k}. \]

By Lemmas 11 and 12 we have

\[ \int_S |S_i(\xi)|^2 dx \ll T^{(e-2^k)(n-\theta_1)} \int_{U_0} |S_i(\xi)|^2 dx \ll T^{(e-2^k)(n-\theta_1) + (2^k - k)n + \theta_1 - \theta} \ll T^{(e-k)n-\theta}. \]
Therefore by Hölder’s inequality,

\[ \int_S S(\xi, T) E(-\nu \xi) d\xi \ll \int_S |S(\xi, T)| d\xi \ll \prod_{i=1}^k \left( \int_S |S_i(\xi)|^i d\xi \right)^{1/i} \ll T^{(e-k)n-\delta}. \]

The lemma is proved.

Proof of Theorem. Set \( T = N(\nu)^{1/kn} \). Our main theorem follows from Lemmas 10 and 13 and (5).

Remarks. 1.) We have not discussed the singular series.

2.) In Ayoub’s proof of his Theorem 5.3 ([1], p. 443), the estimation

\[ \sum_{\lambda \in \mathcal{F}, \lambda \neq \lambda'} S(|\xi|^2 \nu^{-k-1}) \ll T^{k+a-1} N(A)^{-1} h^{-1} N(A)^{-1/n} \]

is used. Since from the definition of \( \xi \) ([1], p. 441), i.e.

\[ N(\max(h|\xi|, t^{-1})) \leq N(A)^{-1} \]

or

\[ \prod_{i=1}^n \max(h|\xi^{(i)}|, t^{-1}) \leq N(A)^{-1}, \]

if \( n > 1 \), i.e. if \( K \) is not rational field it seems that we cannot derive that for all \( i \),

\[ h|\xi^{(i)}| \ll N(A)^{-1/n} \]

or

\[ S(|\xi|) \ll h^{-1} N(A)^{-1/n}. \]

Another thing he used is that \( A(\nu) = A(\nu \eta^k) \) ([1], p. 449). This seems not trivial when \( \phi(\xi) \neq \alpha \xi^k \).
REFERENCES


