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ON A GENERALIZED WARING'S PROBLEM IN ALGEBRAIC NUMBER FIELDS

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1. Introduction.

Let K be a totally real agebraic number field of degree n. Let $K^{(i)}\,(1 \le i \le n)$ be the conjugate fields of K. For $\gamma \in K$, we denote by $\gamma^{(i)}$ $(1 \le i \le n)$ the conjugates of γ and $N(\gamma) = \prod_{i=1}^{n} \gamma^{(i)}$ the norm of γ. Let γ_j $(1 \le j \le n)$ be numbers of K and x_i $(1 \le i \le n)$ be real numbers. We set $\xi = \sum_{j=1}^{n} x_j \gamma_j$ and define $\xi^{(i)} = \sum_{j=1}^{n} x_j \gamma_j^{(i)}$ $(1 \le i \le n)$. We use the notations

$$\|\xi\| = \max_i |\xi^{(i)}|,\ S(\xi) = \sum_{i=1}^{n} \xi^{(i)} \text{ and } E(\xi) = \exp(2\pi i S(\xi)).$$

where $\exp(x) = e^x$. A number γ of K is called totally nonnegative if $\gamma^{(i)} \ge 0$ $(1 \le i \le n)$.

It was Siegel [5,6] who succeeded in dealing with Waring's problem in an arbitrary algebraic number field by his generalized circle method, and obtained the result corresponding to Hardy-Littlewood's estimation on $G(k)$.

Ayoub [1] gave an extension of Siegel's theorem, namely to replace the kth powers by polynomial summands for totally real algebraic number fields. Let ν, α, α_i $(1 \le i \le k-1)$ be nonzero totally nonnegative integers of K and $k \ge 2$. Consider the polynomial

$$\phi(\xi) = \alpha\xi^k + \alpha_{k-1}\,\xi^{k-1} + \cdots + \alpha_1\xi$$

and the equation

$$\nu = \phi(\xi_1) + \cdots + \phi(\xi_s). \tag{1}$$

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Let $A(\nu)$ be the number of solutions of (1) in totally nonnegative integers $\xi_1,\ldots,\xi_s$ satisfying $N(\xi_i)\le N(\nu)^{1/k}\,(1\le i\le s)$. Ayoub established that if $s\ge n(2^k+n)+1$, then

$$A(\nu)=D^{1/2(1-s)}\ \mathfrak{S}(\nu)\Big(\frac{\Gamma(1+\frac{1}{k})}{\Gamma(s/k)}\Big)^n N(\alpha)^{-s/k}\,N(\nu)^{-1+s/k}\ (1+o(1)) \tag{2}$$

where D is the absolute value of the discriminant of K and $\mathfrak{S}(\nu)$ the singular series of (1).

It is our object to give a natural extension of Ayoub's theorem, namely to replace the $\phi(\xi)$ by different polynomials of degree k, and to give a slight improvement on the lower bound for s to $\max(4kn,2^k+1)$. In addition, it seems that there is a gap in his proof of (2) (See Remark in §6).

Consider the polynomials

$$\phi_i(\lambda)=\alpha_{ki}\lambda^{k-1}+\alpha_{k-1,i}\,\lambda^{k-1}+\cdots+\alpha_{1i}\lambda,\ 1\le i\le s \tag{3}$$

and the equation

$$\nu=\phi_1(\lambda_1)+\cdots+\phi_s(\lambda_s), \tag{4}$$

where ν and $\alpha_{ki}\,(1\le i\le s)$ are given nonzero totally nonnegative integers and $\alpha_{ji}\,(1\le j\le k-1,\ 1\le i\le s)$ are integers. Let $B(\nu)$ be the number of solutions of (4) in totally nonnegative integers $\lambda_i\,(1\le i\le s)$ satisfying $N(\lambda_i)\le N(\nu)^{1/k}\,(1\le i\le s)$.

THEOREM. *If $s\ge\max(4kn,2^k+1)$, then*

$$B(\nu)=\mathfrak{S}'J\ N(\nu)^{-1+s/k}\ (1+o(1))$$

where $\mathfrak{S}'$ and J denote the singular series and singular integral of (4) respectively. (See §5).

2. The generalized circle method.

Let $\omega_1,\ldots,\omega_n$ be an integral basis of K and δ the different of K. We can choose a basis $\rho_1,\ldots,\rho_n$ of δ^{-1} such that

$$S(\rho_i,\omega_j)=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases}$$

Set $\xi = x_,\rho_1+\cdots+x_n\rho_n$ and $\eta = y_1\omega_1+\cdots+y_n\omega_n$ where x_i and y_i $(i\le i\le n)$ are real numbers. We denote by $dx=\prod_{i=1}^{n} dx_i$, $dy=\prod_{i=1}^{n} dy_i$ $P(T)$ the set of $\underline{\underline{y}}=(y_1,\ldots,y_n)$ satisfying $0\le\eta^{(i)}\le T$ $(1\le i\le n)$ and $\sum_{\lambda\in P(T)}$ a sum where λ runs over all integers such that $0\le\lambda^{(i)}\le T$ $(1\le i\le n)$.

Let $\mathbb{Q}$ denote the rational number field and U_n the n-dimensional unit cube $\{\underline{\underline{x}}=(x_1,\ldots,x_n):0\le x_i<1(1\le i\le n)\}$. Let h and t be real numbers satisfying $h>2Dt$ and $t>1$. For any $\gamma\in K$, we can determine uniquely two integral ideals $\mathcal{A}_0$, $\mathcal{L}$ such that

$$\gamma\delta=\mathcal{L}/\mathcal{A},\ (\mathcal{A},\mathcal{L})=1.$$

We write $\gamma\to\mathcal{A}$. Let $\Gamma(t)$ be the set consisting of $\gamma=x_1\rho_1+\cdots+x_n\rho_n$ satisfying

$$\underline{\underline{x}}\in U_n: x_i\in\mathbb{Q}(1\le i\le n),\ \gamma\to\mathcal{A}\text{ and }N(\mathcal{A})\le t^n.$$

For every $\gamma\in\Gamma(t)$ subject to $\gamma\to\mathcal{A}$, we define the basis domain B_γ by

$$\underline{\underline{x}}:\underline{\underline{x}}\in U_n,\ \prod_{i=1}^{n}\max(h|\xi^{(i)}-\gamma_0^{(i)}|,t^{-1})\le N(\mathcal{A})^{-1}\text{ for some }\gamma_0\equiv\gamma(\text{mod }\delta^{-1}).$$

We can show that if γ_1 and γ_2 belong to $\Gamma(t)$ and $\gamma_1\neq\gamma_2$, then

$$B_{\gamma_1}\cap B_{\gamma_2}=\phi.$$

(See, Siegel [5,6] or Wang [9]). We set

$$B=\bigcup_{\gamma\in\Gamma(t)}B_r$$

and define the supplementary domain S of B with respect to U_n by

$$S=U_n-B.$$

This division of U_n into B and S depends on (h,t). We shall call this division the Farey division of U_n with respect is (h,t).

Let

$$S_i(\xi,T) = S_i(\xi) = \sum_{\lambda\in P(T)} E(\phi_i(\gamma)\xi) \text{ and } S(\xi,T) = \prod_{i=1}^{s} S_i(\xi).$$

Since for any integer α

$$\int_{U_n} E(\alpha\xi)dx = \begin{cases} 1, & \text{if } \alpha = 0, \\ 0, & \text{if } \alpha \neq 0, \end{cases}$$

the number of solutions of (4) in totally nonnegative integers $\lambda_i (1 \leq i \leq s)$ satisfying $\|\lambda_i\| \leq T (1 \leq i \leq s)$ is equal to

$$Z(\nu) = \int_{U_n} S(\xi,T)E(-\nu\xi)dx = \int_B S(\xi,T)E(-\nu\xi)dx + \int_S S(\xi,T)E(-\nu\xi)dx$$

(5) $$= Z_1 + Z_2,$$

say.

3. Asymptotic expansion for $S_\ell(\xi)$.

Denote by

$$a = \frac{1}{4} + \frac{1}{4kn}, \ t = T^{1-a} \quad \text{and } h = T^{k-1+a}$$

where $T > 1$. For any $\gamma \in \Gamma(t)$, let $\mathcal{L}_\ell = (\alpha_{k\ell}\gamma, \ldots, \alpha_{1\ell}\gamma)$ be the ideal generated by $\alpha_{k\ell}\gamma, \ldots, \alpha_{1\ell}\gamma$ and $\mathcal{A}_\ell$ denote the denominator of $\mathcal{L}_\ell\delta$. We use the notations

$$\xi - \gamma = \varsigma, \ G_\ell(\gamma) = N(\mathcal{A}_\ell)^{-1} \sum_{\lambda(\mathcal{A}_\ell)} E(\phi_\ell(\lambda)\gamma) \text{ and } I_\ell(\varsigma,T) = \int_{P(T)} E(\phi_\ell(\eta)\varsigma)dy,$$

where $\sum_{\lambda(\mathcal{A}_\ell)}$ denotes a sum in which λ runs over a complete residue system mod $\mathcal{A}_\ell$. We use E_n to denote the whole n-dimensional Euclidean space.

LEMMA 1. *(Hua [3]). For any given $\epsilon > 0$,*

$$G_\ell(\gamma) \ll N(\mathcal{A}_\ell)^{\epsilon-1/k},$$

hereafter the constants implicit in "$\ll$" or "O" may depend on ϵ and K. (Note that α_{ij}'s are given integers of K).

LEMMA 2. *(Vinogradov [8]). Let $f(x) = \gamma_k x^k + \cdots + \gamma_1 x$ be a polynomial with real coefficients. Then*

$$\int_0^1 \exp(2\pi i f(x))dx \ll min(1, |\gamma_1|^{-1/k}, \dots, |\gamma_k|^{-1/k}).$$

LEMMA 3. $I_\ell(\varsigma, T) \ll \prod_{i=1}^{n} min(T, |\varsigma^{(i)}|^{-1/k})$.

PROOF: Let $\eta^{(j)} = Tu_j$, $1 \le j \le n$. Then the Jacobian of $y_1, \dots, y_n$ with respect to $u_1, \dots, u_n$ is equal to $D^{-1/2}T^n$. Therefore by Lemma 2,

$$I_\ell(\varsigma, T) = D^{-1/2}T^n \prod_{j=1}^{n} \int_0^1 \exp(2\pi i \varphi_\ell(u_j T)\varsigma^{(j)})du_j \ll \prod_{j=1}^{n} \min(T, |\varsigma^{(j)}|^{-1/k}),$$

and the lemma is proved.

LEMMA 4. *If $s < k$, then*

$$\int_{E_n} \prod_{i=1}^{n} min(T, |\varsigma^{(i)}|^{-1/k})^s dx \ll T^{(s-k)n}.$$

PROOF: Let $\varsigma^{(j)} = v_j, 1 \le j \le n$. Then the Jacobian of $x_1, \dots, x_n$ with respect to $v_1, \dots, v_n$ is $D^{1/2}$. Therefore

$$\int_{E_n} \prod_{j=1}^{n} \min(T, |\varsigma^{(j)}|^{-1/k})^s dx \ll D^{1/2} \prod_{j=1}^{n} \int_0^\infty \min(T^s, v_j^{-s/k})dv_j$$

$$\ll \left(\int_0^{T^{-k}} T^s dv + \int_{T^{-k}}^\infty v^{-s/k}\, dv\right)^n \ll T^{(s-k)n}$$

and the lemma is proved.

LEMMA 5. *Let μ be an integer. Then*

$$\sum_{\substack{\gamma+\mu \in P(T) \\ A|\lambda}} E(\varphi_\ell(\lambda+\mu)\varsigma) + N(A_\ell)^{-1} I_\ell(\varsigma, T) + O(N(A_\ell)^{-1}T^{n-a}).$$

PROOF: The proof is similar to the proof for the sum of $E(\alpha(\lambda+\mu)^k\varsigma)$ (See, Siegel [5,6] or Wang [9]). We give the proof here, for the sake of completeness.

Determine positive numbers $\theta^{(i)}$ $(1 \le i \le n)$ such that

$$\theta^{(i)} \max(h|\varsigma^{(i)}|, t) = D^{1/2n} \prod_{i=1}^{n} \max(h|\varsigma^{(i)}|, t^{-1})^{1/n} N(A_\ell)^{1/n} \tag{6}$$

then

$$\prod_{i=1}^{n} \theta^{(i)} = D^{1/2} N(\mathcal{A}_\ell),$$

and it follows by Minkowski's linear form theorem that there exists $\sigma \in \mathcal{A}_\ell$ such that

$$0 < |\sigma^{(i)}| \le \theta^{(i)}, \; 1 \le i \le n. \tag{7}$$

Hence $\sigma \mathcal{A}_\ell^{-1} = \mathcal{L}_\ell$ is an integral ideal and

$$N(\mathcal{L}_\ell) = |N(\sigma)| N(\mathcal{A}_\ell)^{-1} \le \prod_{i=1}^{n} \theta^{(i)} N(\mathcal{A}_\ell)^{-1} = \sqrt{D}.$$

Therefore $\mathcal{L}_\ell$ belongs to a finite set depending on K only. Let $\sigma_1, \ldots, \sigma_n$ be a basis of $\mathcal{L}_\ell^{-1}$. Then $\mathcal{A}_\ell = \sigma \mathcal{L}_\ell^{-1}$ has a basis

$$\tau_i = \sigma \sigma_i, \; 1 \le i \le n.$$

By (6), (7) we obtain

$$\|\tau_i\| = O(\|\sigma\|) = O(\|\theta\|) = O(t).$$

Expressing λ in terms of τ_i's we have

$$\lambda = g_1 \tau_1 + \cdots + g_n \tau_n,$$

where g_i's are rational integers. Let $U(\lambda)$ denote the unit cube

$$\underline{\underline{s}} : \tau = s_1 \tau_1 + \cdots + s_n \tau_n, \; g_i \le s_i < g_i + 1 (1 \le i \le n)$$

and $ds = \prod_{i=1}^{n} ds_i$. Then by (6) we have

$$\begin{aligned} & E(\varphi_\ell(\lambda + \mu)\varsigma) - \int_{U(\lambda)} E(\varphi_\ell(\tau + \mu)\varsigma) ds \\ & \ll \|(\lambda - \tau)\varsigma\| (\|\tau + \mu\|^{k-1} + \|\lambda + \mu\|^{k-1} + 1) \\ & \ll \|\theta\varsigma\| T^{k-1} \ll h^{-1} T^{k-1} \ll T^{-a}. \end{aligned}$$

Since the number of integers λ with $\mathcal{A}_\ell \mid \lambda$ and $\lambda + \mu \in P(T)$ is $O(N(\mathcal{A}_\ell)^{-1} T^n)$, we have

$$\begin{aligned} \sum_{\substack{\lambda+\mu \in P(T) \\ \mathcal{A}_\ell | \lambda}} E(\varphi_\ell(\lambda + \mu)\varsigma) = & \sum_{\substack{\lambda+\mu \in P(T) \\ \mathcal{A}_\ell | \lambda}} \int_{U(\lambda)} E(\varphi_\ell(\tau + \mu)\varsigma) ds \\ & + O(N(\mathcal{A}_\ell)^{-1} T^{n-a}) \end{aligned}$$

$$= \int_{0 \le \mu^{(i)} + \tau^{(i)} \le T} \cdots \int E(\varphi_\ell(\tau+\mu)\varsigma)ds + O(N(\mathcal{A}_\ell)^{-1} T^{n-a}).$$

Let $\tau + \mu = \eta$. Since the Jacobian of s_i's with respect to y_i's is $D^{1/2}|\det(\tau_j^{(i)})|^{-1} = N(\mathcal{A}_\ell)^{-1}$, the lemma follows.

LEMMA 6. $S_\ell(\xi) = G_\ell(\gamma) I_\ell(\varsigma, T) + O(T^{n-a})$.

PROOF: By Lemma 5,

$$\begin{aligned} S_\ell(\xi) &= \sum_{\mu(\mathcal{A}_\ell)} E(\phi_\ell(\mu)\gamma) \sum_{\substack{\lambda+\mu\in P(T)\\ \mathcal{A}_\ell|\lambda}} E(\phi(\lambda+\mu)\varsigma) \\ &= \sum_{\mu(\mathcal{A}_\ell)} E(\phi_\ell(\mu)\gamma)(N(\mathcal{A}_\ell)^{-1} I_\ell(\varsigma,T) + O(N(\mathcal{A}_\ell)^{-1}T^{n-a})) \\ &= G_\ell(\gamma) I_\ell(\varsigma,T) + O(T^{n-a}). \end{aligned}$$

The lemma is proved.

4. Basic domains.

Set $G(\gamma) = \prod_{i=1}^{s} G_i(\gamma)$, $S(\xi,T) = \prod_{i=1}^{s} S_i(\xi)$ and $I(\varsigma,T) = \prod_{i=1}^{s} I_i(\varsigma,T)$.

LEMMA 7. *If $s \ge 4kn$, then*

$$\int_B S(\xi,T)E(-\nu\xi)dx = \sum_{\gamma\in\Gamma(t)} G(\gamma)E(-\nu\gamma)\int_{B_\gamma} I(\varsigma,T)E(-\nu\varsigma)dx + 0(T^{(s-k)n-a}).$$

PROOF: Let $\gamma \to \mathcal{A}$. Since $\mathcal{A}_i|\mathcal{A}|\alpha_{k1}\ldots\alpha_{1i}\mathcal{A}_i$, we have

$$N(\mathcal{A}_i)^{-1} \ll N(\mathcal{A})^{-1} \ll N(\mathcal{A}_i)^{-1},\ 1 \le i \le s.$$

By Lemma 1, 3 and 6, we have

$$\begin{aligned} S(\xi,T) &= G(\gamma)I(\varsigma,T) + O\Big(T^{n-a}\max_j\big(T^{n-a}, |G_j(\gamma)I_j(\varsigma,T)|\big)^{s-1}\Big) \\ &= G(\gamma)I(\varsigma,T) + O(T^{(n-a)s}) + O(T^{n-a}(N(\mathcal{A})^{1/k+\epsilon/s}\prod_{i=1}^{n}\min(T,|\varsigma^{(i)}|^{-1/k})^{s-1}). \end{aligned}$$

Since the number of $\gamma \in \Gamma(t)$ such that $\gamma \to \mathcal{A}$ is $O(N(\mathcal{A}))$ and the number $\mathcal{A}$ such that $N(\mathcal{A}) = m$ is $O(m^\epsilon)$, we have

$$\sum_{\gamma\in\Gamma(t)} N(\mathcal{A})^{-\frac{s-1}{k}+\varepsilon} \ll \sum_{m\le t^n} m^{-\frac{4kn-1}{k}+1+2\varepsilon} \ll 1, \tag{8}$$

and by Lemma 4 and $a = \frac{1}{4} + \frac{1}{4k^n}$,

$$\int_B S(\xi,T)E(-\nu\xi)dx = \sum_{\gamma\in\Gamma(t)} G(\gamma)E(-\nu\gamma)\int_{B_\gamma} I(\varsigma,T)E(-\nu\varsigma)dx$$

$$+ O(T^{(n-a)s}) + O\Big(T^{n-a}\sum_{\gamma\in\Gamma(t)} N(\mathcal{A})^{-\frac{s-1}{k}+\varepsilon}\int_{E_n}\prod_{i=1}^{n}\min(T, |\varsigma^{(i)}|^{-1/k})^{s-1}dx\Big)$$

$$= \sum_{\gamma\in\Gamma(t)} G(\gamma)E(-\nu\gamma)\int_{B_\gamma} I(\varsigma,T)E(-\nu\varsigma)dx + O(T^{(s-k)n-a})$$

The lemma is proved.

LEMMA 8. *If $s \ge 4kn$, then*

$$\int_B S(\xi,T)E(-\nu\xi)dx = \sum_{\gamma\in\Gamma(t)} G(\gamma)E(-\nu\gamma)\int_{E_n} I(\varsigma,T)E(-\nu\varsigma)dx + O(T^{(s-k)n-a}).$$

PROOF: By Lemma 7, it suffices to show that

$$W = \sum_{\gamma\in\Gamma(t)} G(\gamma)E(-\nu\gamma)\int_{E_n-B_\gamma} I(\varsigma,T)E(-\nu\varsigma)dx = O(T^{(s-k)n-a}).$$

If $\underline{\underline{x}} \in E_n - B_\gamma$, then there is at least one index i such that $h|\varsigma^{(i)}| > N(\mathcal{A})^{1/n}$. Hence by Lemma 3

$$\int_{E_n-B_\gamma} I(\varsigma,T)E(-\nu\varsigma)dx \ll \int_{E_n-B_\gamma}\prod_{i=1}^{n}\min(T^s, |\varsigma^{(i)}|^{-s/k})dx$$

$$\ll \int_{h^{-1}N(\mathcal{A})^{-1/n}}^{\infty} u^{s/k}du\Big(\int_0^\infty \min(T^s, v^{-s/k})dv\Big)^{n-1}$$

$$\ll h^{s/k-1} N(\mathcal{A})^{\frac{1}{n}(\frac{s}{k}-1)} T^{(s-k)(n-1)} .$$

By Lemma 1, we have

$$\sum_{\gamma\in\Gamma(t)} G(\gamma)N(\mathcal{A})^{\frac{1}{n}(\frac{s}{k}-1)} \ll \sum_{\gamma\in\Gamma(t)} N(\mathcal{A})^{\varepsilon-\frac{s}{k}+\frac{s}{k}-1} \ll t^{2n}.$$

Therefore

$$W \ll T^{(s-k)n-(\frac{s}{k}-1)(1-a)+2(1-a)n} \ll T^{(s-k)n-a} .$$

The lemma is proved.

5. Some more lemmas.

Set

$$J_i(\varsigma,T)=\int_{P(T)}E(\alpha_{ki}\eta^k\varsigma)dy(1\le i\le s)\quad\text{and}\quad J(\varsigma,T)=\prod_{i=1}^{s}J_i(s,T).$$

Since

$$E(\phi_i(\eta)\varsigma)-E(\alpha_{ki}\eta^k\varsigma)\ll S(|\varsigma|)T^{k-1},$$

we have for $\theta=-k+\frac{1}{n+1}$,

$$\int_{E_n}(I(\varsigma,T)-J(\varsigma,T))E(-\nu\varsigma)dx\ll\int_{\|\varsigma\|\le T^\theta}|I(\varsigma,T)-J(\varsigma,T)|dx$$

$$+\sum_{i=1}^{n}\int_{|\varsigma^{(i)}|>T^\theta}|I(\varsigma,T)-J(\varsigma,T)|dv=\Sigma_1+\Sigma_2,\text{ say.}$$

By Lemma 3, we have

$$\sum_1\ll\int_0^{T^\theta}\cdots\int_0^{T^\theta}T^{k-1}\sum_{j=1}^{n}v_j\prod_{i=1}^{n}\min(T,v_i^{1/k})^{s-1}dv_1\dots dv_n$$

$$\ll T^{(s-1)n+k-1+(n+1)\theta}\quad\ll T^{(s-k)n-n}$$

and for $s\ge 4kn$,

$$\Sigma_2\ll\int_{T^\theta}^{\infty}r^{-s/k}dv\int_{E_{n-1}}\prod_{j=1}^{n-1}\min(T,v_j^{-1/k})^s dv_1\dots dv_{n-1}$$

$$\ll T^{(-s/k+1)\theta+(s-k)(n-1)}\quad\ll T^{(s-k)n-(s/k-1)\frac{1}{n+1}}\quad\ll T^{(s-k)n-1}.$$

By (8) and Lemmas 1 and 8, we have

LEMMA 9. *If $s\ge 4kn$, then*

$$\int_B S(\xi,T)E(-\nu\xi)dx=\sum_{\gamma\in\Gamma(t)}G(\gamma)E(-\nu\gamma)\int_{E_n}J(\varsigma,T)E(-\nu\varsigma)dx$$

$$+0(T^{(s-k)n-a}).$$

Let $\eta'=y_1'\omega_1+\cdots+y_n'\omega_n$, $\varsigma'=x_1'\rho_1+\cdots+x_n'\rho_n$, $dy'=dy_1'\dots dy_n'$, $dx'=dx_1'\dots dx_n'$, $\eta=T\eta'$ and $\varsigma=T^{-k}\varsigma'$. The Jacobian of $y_1,\dots,y_n$ and $x_1,\dots,x_n$ with respect to $y'_n, \dots, y'_n$ and $x'_1, \dots, x'_n$ are T^n and T^{-kn} respectively. Then

$$\alpha_{ki}\eta^k\varsigma = \alpha_{ki}\eta'^k\varsigma'.$$

Write η' and ς' by η and ς again. Then

$$\int_{E_n} J(\varsigma, T)E(-\nu\varsigma)dx = T^{(s-k)n}\, J(\mu), \tag{9}$$

where $\mu = \nu T^{-k}$ and

$$J(\mu) = J = \int_{E_n} \Big(\prod_{i=1}^{s} J_i(\varsigma, 1)\Big)E(-\mu\varsigma)dx.$$

J is called the singular integral which is absolutely convergent by Lemma 3 if $s > k$. Similar to Siegal [5,6] or Tatuzawa [7], Wang [9], we have

$$J = D^{\frac{1}{2}(1-s)}\, k^{-sn}\, N(\alpha_{k1}\dots\alpha_{ks})^{1/k} \prod_{\ell=1}^{n} F_\ell,$$

where

$$F_\ell = \int_{W_\ell} \prod_{i=1}^{s} w^{\frac{1}{s}-1}\, dw$$

in which $dw = \prod_{i=1}^{s-1} dw_i$ and W_ℓ denotes the domain

$$0 \le \omega_i \le \alpha_{ki}^{(\ell)} \ (1 \le i \le s),\ \mu^{(\ell)} - \omega_1 - \dots - \omega_s = 0.$$

By Lemma 1 we have for $s \ge 4kn$

$$\sum_{\substack{\gamma\to A\\ N(A)>t^n}} G(\gamma)E(-\nu\gamma) \ll \sum_{\substack{\gamma\to A\\ N(A)>t^n}} N(A)^{-\frac{s}{k}+\epsilon} \ll T^{-(1-a)n}\ .$$

and thus

$$\sum_{\gamma\in\Gamma(t)} G(\gamma)E(-\nu\gamma) = \mathfrak{S}' + O(T^{-(1-a)n}) \tag{10}$$

where

$$\mathfrak{S}' = \sum_{\gamma} G(\gamma)E(-\nu\gamma)$$

in which γ runs over a complete residue system of $(A\delta)^{-1}$, mod δ^{-1}, $\mathfrak{S}'$ is called the singular series.

By Lemma 9. (9), (10) and (11), we have

LEMMA 10. *If $s \ge 4k^n$, then*

$$\int_B \delta(\xi, T)E(-\nu\xi)dx = \mathfrak{S}' J\, T^{(s-k)n}\,(1+o(1)).$$

6. Proof of the theorem.

Let θ be a number satisfying $0 < \theta < (1-a)2^{1-k}$.

LEMMA 11. *If $\underline{x} \in S$, then*

$$S_i(\xi) \ll T^{n-\theta}$$

where the constant implicit in $\ll$ depends on θ.

This can be proved a a Siegel's lemma [5,6] on the diophantine approximation for $\xi(\underline{x} \in S)$ and a theorem of Mitsui [5]. See, Tatuzawa [7], p. 54.

LEMMA 12. *If $1 \le j \le k$, then*

$$\int_{U_n} |S_i(\xi)|^{2^j} dx \ll T^{(2^j-j)n+\varepsilon}\ .$$

This is a generalization of Hua's inequality [2] in algebraic number fields. See, Ayoub [1], p. 447.

LEMMA 13. *If $s \ge 2^k + 1$, then*

$$\int_S S(\xi, T)E(-\nu\xi)dx \ll T^{(s-k)-\theta}\ .$$

PROOF: Put

$$\theta_1 = \frac{\theta}{2} + \frac{1-a}{2^k}.$$

Then

$$\theta < \theta_1 < (1-a)2^{1-k}.$$

By Lemmas 11 and 12 we have

$$\int_S |S_i(\xi)|^s dx \ll T^{(s-2^k)(n-\theta_1)} \int_{U_n} |S_i(\xi)|^{2^k} dx$$
$$\ll T^{(s-2^k)(n-\theta_1)+(2^k-k)n+\theta_1-\theta} \quad \ll T^{(s-k)n-\theta}\ .$$

Therefore by Hõlder's inequality,

$$\int_S S(\xi,T)E(-\nu\xi)dx \ll \int_S |S(\xi,T)|dx \ll \prod_{i=1}^{s}\Big(\int_S |S_i(\xi)|^s dx\Big)^{1/s}$$
$$\ll T^{(s-k)n-\theta} \ .$$

The lemma is proved.

Proof of Theorem. Set $T = N(\nu)^{1/kn}$. Out main theorem follows from Lemmas 10 and 13 and (5).

<u>Remarks</u>. 1.) We have not discussed the singular series.

2.) In Ayoub's proof of his Theorem 5.3 ([1], p. 443), the estimation

$$\sum_{\substack{\lambda+\mu\in\mathcal{F}\\ \mathcal{A}|\lambda}} S(|\varsigma|T^{k-1}) \ll T^{k+n-1}\, N(\mathcal{A})^{-1} h^{-1} N(\mathcal{A})^{-1/n}$$

is used. Since from the definition of ς ([1], p. 441), i.e.

$$N(\max(h|\varsigma|, t^{-1}) \le N(\mathcal{A})^{-1}$$

or

$$\prod_{i=1}^{n} \max(h|\varsigma^{(i)}|, t^{-1}) \le N(\mathcal{A})^{-1},$$

if $n > 1$, i.e. if K is not rational field it seems that we cannot derive that for all i,

$$h|\varsigma^{(i)}| \ll N(\mathcal{A})^{-1/n}$$

or

$$S(|\varsigma|) \ll h^{-1} N(\mathcal{A})^{-1/n}\,.$$

Another thing he used is that $A(\nu) = A(\nu\eta^k)$ ([1], p. 449). This seems not trivial when $\phi(\xi) \neq \alpha\xi^k$.

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