THE DIVISOR PROBLEM FOR \((k, r)\)-INTEGERS

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1. Introduction

Let \(k\) and \(r\) be fixed integers such that \(1 < r < k\). It is well-known that a positive integer is called \(r\)-free if it is not divisible by the \(r\)-th power of any integer > 1. We call a positive integer \(n\), a \((k, r)\)-integer, if \(n\) is of the form \(n = a^k b\), where \(a\) is a positive integer and \(b\) is a \(r\)-free integer. In the limiting case, when \(k\) becomes infinite, a \((k, r)\)-integer becomes a \(r\)-free integer and so one might consider the \((k, r)\) integers as generalized \(r\)-free integers.

It has been shown by one of the authors and V. Siva Rama Prasad [4] that if \(\tau_{(r)}(n)\) denotes the number of \(r\)-free divisors of \(n\), then for \(x \geq 3\),

\[
\sum_{n \leq x} \tau_{(r)}(n) = \frac{x}{\zeta(r)} \left( \log x + 2\gamma - 1 - \frac{r \zeta'(r)}{\zeta(r)} \right) + \Delta_r(x),
\]

where \(\Delta_r(x) = O(x^{1/r} \delta(x))\) or \(O(x^\varepsilon)\), according as \(r = 2, 3\) or \(r \geq 4\);
\(\delta(x) = \exp \{ - A \log^{3/5} x (\log \log x)^{-1/5} \}\), \(A\) being a positive constant and \(\varepsilon\) is the number which appears in the Dirichlet divisor problem

\[
\sum_{n \leq x} \tau(n) = x(\log x + 2\gamma - 1) + O(x^\varepsilon),
\]

where \(\tau(n)\) is the number of divisors of \(n\).

It is known that \(\frac{1}{2} < \varepsilon < \frac{1}{3}\) (cf. [1], p. 272). The best result yet proved has been obtained recently by Kolesnik [2], who proved that the error term in (1.2) is \(O(x^{(12/37)^+})\), for any \(\varepsilon > 0\). There is a conjecture that \(\varepsilon = \frac{1}{4} + \varepsilon\). In the formula (1.1), \(\zeta(s)\), denotes the Riemann Zeta function and \(\zeta'(s)\) its derivative and \(\gamma\) is Euler's constant.

It has also been shown in [4] on the assumption of the Riemann hypothesis that \(\Delta_3(x) = O(x^{(5 - 4\varepsilon)/(5 - 4\varepsilon)} \omega(x))\), \(\Delta_3(x) = O(x^{(7 - 6\varepsilon)/(7 - 6\varepsilon)} \omega(x))\) and \(\Delta_4(x) = O(x^{\varepsilon})\)

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2 On leave from Andhra University, Waltair, India.
for \( r \geq 4 \), where \( \omega(x) = \exp \{ A \log x (\log \log x)^{-1} \} \), \( A \) being a positive constant. For earlier (weaker) estimations of \( \Delta_r(x) \) by various authors, we refer to the bibliography given in [4].

Let us call a divisor \( d \) of a positive integer \( n \), an \((k,r)\)-divisor of \( n \) if \( d \) is a \((k,r)\)-integer. Let \( \tau_{(k,r)}(n) \) denote the number of \((k,r)\)-divisors of \( n \). The object of this paper is to prove the following:

**Theorem 1.** For \( 1 < r < k \) and \( x \geq 3 \),

\[
\sum_{n \leq x} \tau_{(k,r)}(n) = \frac{\zeta(k)x}{\zeta(r)} \left( \log x + 2\gamma - 1 - \frac{r\zeta'(r)}{\zeta(r)} + \frac{k\zeta'(k)}{\zeta(k)} \right) + \Delta_{k,r}(x),
\]

where \( \Delta_{k,r}(x) = O(x^{1/r}\delta(x)) \) or \( O(x^r) \), according as \( r = 2, 3 \) or \( 4 \leq r < k \), the \( \theta \)-estimates being uniform in \( k \); \( \delta(x) = \exp \{ -B \log^{3/5} x (\log \log x)^{-1/5} \} \), \( B \) being a positive constant and \( \alpha \) is the number which appears in (1.2).

**Theorem 2.** If the Riemann hypothesis is true, then the error term \( \Delta_{k,r}(x) \) in (1.3) has the following improved \( \theta \)-estimates:

\[
\Delta_{3,2}(x) = O(x^{5/11} \omega(x)), \quad \Delta_{4,2}(x) = O(x^{(2-s)/(5-6s)} \omega(x))
\]

for \( k \geq 4 \), \( \Delta_{k,3}(x) = O(x^{(2-s)/(7-6s)} \omega(x)) \) for \( k \geq 4 \) and \( \Delta_{r}(x) = O(x^s) \) for \( 4 \leq r < k \); where the \( \theta \)-estimates are uniform in \( k \) and \( \omega(x) = \exp \{ A \log x (\log \log x)^{-1} \} \), \( A \) being a positive constant and \( \alpha \) is given by (1.2).

It may be noted that in the limiting case when \( k \to \infty \), formula (1.3) coincides with (1.1) and the \( \theta \)-estimates of \( \Delta_r(x) = \Delta_{\infty,r}(x) \) obtained in [4] follow as a particular case.

### 2. Prerequisites

In this section we prove some lemmas which are needed in the proofs of Theorem 1 and 2. Throughout the following, \( x \) denotes a real variable \( \geq 3 \). The following elementary estimates are well-known:

\[
\sum_{n \leq x} \frac{1}{n^s} = O(x^{1-s}) \text{ if } 0 \leq s < 1.
\]

\[
\sum_{n > x} \frac{1}{n^s} = \zeta(s) - \sum_{n \leq x} \frac{1}{n^s} = 0 \left( \frac{1}{x^{s-1}} \right) \text{ if } s > 1.
\]

\[
\sum_{n < x} \frac{\log n}{n^s} = -\zeta'(s) - \sum_{n \leq x} \frac{\log n}{n^s} = 0 \left( \frac{\log x}{x^{s-1}} \right) \text{ if } s > 1.
\]

**Lemma 2.1** (cf., [6]; Satz 3, p. 191).

\[
M(x) = \sum_{n \leq x} \mu(n) = O(x\delta(x)),
\]
where
\[ \delta(x) = \exp \{-A \log^{3/5} x \log \log x \}^{1/5} \],
\[ A \text{ being a positive constant.} \]

\textbf{Lemma 2.2 (cf. [4] Lemma 2.2).} For any \( s > 1 \),
\[ \sum_{n \leq x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O \left( \frac{\delta(x)}{x^{1+\epsilon}} \right). \]

\textbf{Lemma 2.3 (cf. [4], Lemma 2.3).} For any \( s > 1 \),
\[ \sum_{n \leq x} \frac{\mu(n) \log n}{n^s} = \frac{\zeta'(s)}{\zeta^2(s)} + O \left( \frac{\delta(x) \log x}{x^{s-1}} \right). \]

\textbf{Lemma 2.4 (cf. [5], Theorem 14-26 (A), p. 316).} If the Riemann hypothesis is true, then
\[ M(x) = \sum_{n \leq x} \mu(n) = O(x^{1/2} \omega(x)), \]
where
\[ \omega(x) = \exp \{ A \log x (\log \log x)^{-1} \}, \]
\[ A \text{ being a positive constant.} \]

\textbf{Lemma 2.5 (cf. [4], Lemma 2.5).} If the Riemann hypothesis is true, then for any \( s > 1 \),
\[ \sum_{n \leq x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O(x^{1-s} \omega(x)). \]

\textbf{Lemma 2.6 (cf. [4], Lemma 2.6).} If the Riemann hypothesis is true, then for any \( s > 1 \),
\[ \sum_{n \leq x} \frac{\mu(n) \log n}{n^s} = \frac{\zeta'(s)}{\zeta^2(s)} + O(x^{1-s} \omega(x) \log x). \]

\textbf{Lemma 2.7 (cf. [3], Lemma 2.6).} If \( q_{k,r}(n) \) denotes the characteristic function of the set of \((k,r)\)-integers, that is, \( q_{k,r}(n) = 1 \) or 0 according as \( n \) is or is not a \((k,r)\)-integer, then
\[ q_{k,r}(n) = \sum_{a^b c^r = n} \mu(b). \]

\textbf{Lemma 2.8.} \( \tau_{k,r}(n) = \sum_{a^b c^r = n} \mu(b) \).

\textbf{Proof.} We have \( \tau_{k,r}(n) = \sum_{d \mid n} q_{k,r}(d) \), so that by (2.12),
\[ \tau_{k,r}(n) = \sum_{d \mid n} \sum_{a^b c^r \mid d} \mu(b) = \sum_{a^b c^r \mid n} \mu(b) \]
\[ \sum_{a^k b^r c = n} \mu(b) \frac{\sum_{c \in (a^k b^r)} 1}{\sum_{a^k b^r | n}} = \sum_{a^k b^r c = n} \mu(b) \tau \left( \frac{n}{a^k b^r} \right) \]

\[ = \sum_{a^k b^r c = n} \mu(b) \tau(c). \]

Hence Lemma 2.8 follows.

**Lemma 2.9.** For \( k \geq 3 \),
\[ \sum_{a^k c \leq x} \tau(c) = \zeta(k) x \left( \log x + 2\gamma - 1 + \frac{k\zeta'(k)}{\zeta(k)} \right) + R_k(x), \]
where
\[ R_k(x) = O(x^{\frac{1}{k}} \log x) \text{ or } O(x^\gamma), \]
according as \( k = 3 \) or \( k \geq 4 \), where the second \( O \)-estimate is uniform in \( k \).

**Proof.** We have by (1.2), (2.2) and (2.3),
\[ \sum_{a^k c \leq x} \tau(c) = \sum_{a \leq \sqrt{x}} \sum_{c \leq x/ak} \tau(c) \]
\[ = \sum_{a \leq \sqrt{x}} \sum_{c \leq x/ak} \left( \frac{x}{a^k} \left( \log \frac{x}{a^k} + 2\gamma - 1 \right) + O \left( \frac{x}{a^k} \right) \right) \]
\[ = x(\log x + 2\gamma - 1) \sum_{a \leq \sqrt{x}} \frac{1}{a^k} - kx \sum_{a \leq \sqrt{x}} \frac{\log a}{a^k} + O \left( x^\gamma \sum_{a \leq \sqrt{x}} a^{-ka} \right) \]
\[ = x(\log x + 2\gamma - 1) \left( \zeta(k) + O(x^{-1+1/k}) \right) - kx \left( - \zeta'(k) \right) \]
\[ + O \left( \frac{\log x}{x^{1-1/k}} \right) + O \left( x^\gamma \sum_{a \leq \sqrt{x}} a^{-ka} \right) \]
\[ = \zeta(k) x \left( \log x + 2\gamma - 1 + \frac{k\zeta'(k)}{\zeta(k)} \right) + O(x^{1/\gamma} \log x) + O \left( x^\gamma \sum_{a \leq \sqrt{x}} a^{-ka} \right). \]

Since \( \frac{1}{k} < x < \frac{1}{k}, \) we have \( kx \leq 1 \) according as \( k = 3 \) or \( k \geq 4 \). Hence, by (2.1) and (2.2), the last \( O \)-term in the above is \( O(x^\gamma) \) or \( O(\zeta(kx)x^\gamma) = O(\zeta(4x)x^\gamma) = O(x^\gamma) \), uniformly in \( k \), according as \( k = 3 \) or \( k \geq 4 \). Hence Lemma 2.9 follows.

3. Proof of Theorem 1

By Lemma 2.8, we have
\[ \tau_{(k, r)}(n) = \sum_{a^k b^r c = n} \mu(b) \tau(c). \]
Hence
\[ \sum_{n \leq x} \tau_{(k,r)}(n) = \sum_{n \leq x} \sum_{a^{b}b^{c} = n} \mu(b) \tau(c) = \sum_{a^{b}b^{c} \leq x} \mu(b) \tau(c), \]

where the summation on the right being taken over all ordered triads \((a, b, c)\) such that \(a^{b}b^{c} \leq x\).

Let \(z = x^{1/r}\). Further, let \(0 < \rho = \rho(x) < 1\), where the function \(\rho(x)\) will be suitably chosen later.

Now, if \(a^{b}b^{c} \leq x\), then both \(b > \rho z\) and \(a^{c} > \rho^{-r}\) can not simultaneously hold good. Hence from (3.1), we have

\[ \sum_{n \leq x} \tau_{(k,r)}(n) = \sum_{a^{b}b^{c} \leq x} \mu(b) \tau(c) + \sum_{a^{b}b^{c} \leq x} \mu(b) \tau(c) - \sum_{a^{b}b^{c} \leq x} \mu(b) \tau(c) \]

\[ = S_1 + S_2 - S_3, \text{ say.} \]

By (2.13), we have

\[ S_1 = \sum_{b \leq \rho z} \mu(b) \tau(c) = \sum_{b \leq \rho z} \mu(b) \sum_{a^{c} \leq x/b^{r}} \tau(c) \]

\[ = \sum_{b \leq \rho z} \mu(b) \left( \zeta(k) \left( \frac{x}{b^{r}} \right) \log \left( \frac{x}{b^{r}} \right) + \frac{k_{r}^{(c)}(k)}{\zeta(k)} \right) \]

\[ + R_{k} \left( \frac{x}{b^{r}} \right) \]

\[ = \zeta(k)x \left( \log x + 2 \gamma - 1 + \frac{k_{r}^{(c)}(k)}{\zeta(k)} \right) \sum_{b \leq \rho z} \frac{\mu(b)}{b^{r}} \]

\[ - \zeta(k)rx \sum_{b \leq \rho z} \frac{\mu(b) \log b}{b^{r}} + E_{k,r}(x), \]

where

\[ E_{k,r}(x) = \sum_{b \leq \rho z} \mu(b) R_{k} \left( \frac{x}{b^{r}} \right). \]

Hence by (3.3), (2.6) and (2.7), we have

\[ S_1 = \zeta(k)x \left( \log x + 2 \gamma - 1 + \frac{k_{r}^{(c)}(k)}{\zeta(k)} \right) \left[ \frac{1}{\zeta(r)} + O\left( \frac{\delta(\rho z)}{(\rho z)^{r-1}} \right) \right] \]

\[ - \zeta(k)rx \left[ \frac{\zeta'(r)}{\zeta(r)} + O\left( \frac{\delta(\rho z) \log(\rho z)}{(\rho z)^{r-1}} \right) \right] + E_{k,r}(x) \]

\[ = \frac{\zeta(k)x}{\zeta(r)} \left( \log x + 2 \gamma - 1 - \frac{r \zeta'(r)}{\zeta(r)} + \frac{k_{r}^{(c)}(k)}{\zeta(k)} \right) \]

\[ + O(\zeta(k)\rho^{1-r}z \delta(\rho z) \log z) + E_{k,r}(x). \]

By (2.14) and (3.4), we have

\[ E_{k,r}(x) = O \left( \sum_{b \leq \rho z} \frac{x^{b}}{b^{r+3}} \log \left( \frac{x}{b^{r}} \right) \right) \text{ or } O \left( \sum_{b \leq \rho z} \frac{x^{b}}{b^{r+3}} \right), \]
according as \( k = 3 \) or \( k \geq 4 \). Since \( 1 < r < k \), we have \( r = 2 \), when \( k = 3 \) and since \( \frac{1}{2} < x < \frac{1}{2} \), we have by (2.1) and (2.2), the following 0-estimates:

\[
\begin{align*}
E_{a,2}(x) &= O(\rho^{1/3} x^{1/2} \log x) \\
E_{a,r}(x) &= O(\rho^{1-r} x^r) \\
E_{k,r}(x) &= O(\rho^{1-r} x^r) \text{ or } O(x^r),
\end{align*}
\]

(3.6)

where the 0-estimates are uniform in \( k \). We have

\[
S_2 = \sum_{a^b c \leq x, a^b \leq \rho^{-r}} \mu(b) \tau(c) = \sum_{a^b \leq \rho^{-r}} \tau(c) \sum_{b \leq \sqrt{x/(a^b c)}} \mu(b)
\]

\[
= \sum_{a^b \leq \rho^{-r}} \tau(c) M \left( \frac{\sqrt{x}}{a^b c} \right)
\]

\[
= 0 \left( x^{1/r} \sum_{a^b \leq \rho^{-r}} \tau(c) a^{-k/r} c^{-1/r} \delta \left( \frac{\sqrt{x}}{a^b c} \right) \right).
\]

by (2.4). Since \( \delta(x) \) is monotonic decreasing and \( \sqrt{x} \leq z \), we have

\[
\delta \left( \frac{\sqrt{x}}{a^b c} \right) \leq \delta(\rho z) . \text{ Also, by (2.1), (2.2) and (1.2),}
\]

\[
\sum_{a^b \leq \rho^{-r}} \tau(c) a^{-k/r} c^{-1/r} = \sum_{a \leq \rho^{-r}} a^{-k/r} \sum_{c \leq \rho^{-r} a^{-k}} \tau(c) c^{-1/r}
\]

\[
= O \left( \sum_{a \leq \rho^{-r}} a^{-k/r} \rho^{-r} a^{-k} \log(\rho^{-r} a^{-k}) \right)
\]

\[
= O \left( \rho^{1-r} \log \left( \frac{1}{\rho} \right) \sum_{a \leq \rho^{-r}} a^{-k} \right)
\]

\[
= O \left( \zeta(k) \rho^{1-r} \log \left( \frac{1}{\rho} \right) \right).
\]

Hence

(3.7)

\[
S_2 = O \left( \zeta(k) \rho^{1-r} \log \left( \frac{1}{\rho} \right) \delta(\rho z) \right).
\]

Further, we have by (2.4) and (2.13),

(3.8)

\[
S_3 = \sum_{b^c \leq \rho z, a^c \leq \rho^{-r}} \mu(b) \tau(c) = \sum_{b \leq \rho z} \mu(b) \sum_{a^c \leq \rho^{-r}} \tau(c)
\]

\[
= M(\rho z) \sum_{a^c \leq \rho^{-r}} \tau(c)
\]

\[
= O(\rho z \delta(\rho z) \zeta(k) \rho^{-r} \log(\rho^{-r})).
\]
\[ = O \left( \zeta(k) \rho^{1-r} z \delta(\rho z) \log \left( \frac{1}{\rho} \right) \right). \]

Hence by (3.2), (3.5), (3.7) and (3.8)
\begin{equation}
\sum_{n \in \pi} \tau_{k,r}(n) = \frac{\zeta(k)x}{\zeta(r)} \left( \log x + 2\gamma - 1 - \frac{\zeta'(r)}{\zeta(r)} + \frac{k\zeta'(k)}{\zeta(k)} \right) + 0(\zeta(k) \rho^{1-r} z \delta(\rho z) \log z) + 0 \left( \zeta(k) \rho^{1-r} \delta(\rho z) \log \left( \frac{1}{\rho} \right) \right) + E_{k,r}(x).
\end{equation}

Now, we choose,
\begin{equation}
\rho = \rho(x) = \left( \delta \left( x^{1/2r} \right) \right)^{1/r},
\end{equation}
and write
\begin{equation}
f(x) = \log^{3/5} \left( x^{1/2r} \right) \left( \log \log \left( x^{1/2r} \right) \right)^{-1/5}
= \left( \frac{1}{2r} \right)^{3/5} U^{3/5} (V - \log 2) \left( \frac{V}{2} \right)^{-1/5},
\end{equation}
where \( U = \log x \) and \( V = \log \log x \).

(3.12) For \( V \geq 2 \log 2r \), that is, \( U \geq 4r^2, \ x \geq \exp(4r^2) \), we have
\[ V^{-1/5} \leq (V - \log 2r)^{-1/5} \leq \left( \frac{V}{2} \right)^{-1/5} \]
and therefore
\begin{equation}
\frac{1}{2r} r^{-3/5} U^{3/5} V^{-1/5} \leq f(x) \leq r^{-3/5} U^{3/5} V^{-1/5}.
\end{equation}

(3.14) We assume without loss of generality that the constant \( A \) in (2.5) is less than 1.

By (3.10), (2.5) and (3.11), we have
\begin{equation}
\rho = \exp \left( -A \frac{f(x)}{r} \right).
\end{equation}
By (3.12), we have
\[ r^{-8/5} U^{3/5} V^{-1/5} \leq \frac{U}{2r}. \]
Hence, by (3.13), (3.14), (3.15) and the above,
\[ \rho \geq \exp \left( -A r^{-8/5} U^{3/5} V^{-1/5} \right) \geq \exp \left( - r^{-8/5} U^{3/5} V^{-1/5} \right) \]
\[ \geq \exp \left( - \frac{U}{2r} \right) = \exp \left( - \frac{\log x}{2r} \right), \]
so that \( \rho \geq x^{-(1/2r)} \).
(3.16) \( \log \left( \frac{1}{\rho} \right) \leq \log(\sqrt{2}) = o(\log x) \) and \( \rho z \geq x^{1/(2r)} \).

Since \( \delta(x) \) is monotonic decreasing, we have \( \delta(\rho z) \leq \delta(x^{1/(2r)}) = \rho' \), by (3.10), so that by (3.13) and (3.15), we have

\[
(3.17) \quad \rho^{1-r} \delta(\rho z) \leq \rho \leq \exp \left\{ - \frac{A}{2} r^{-8/5} U^{3/5} V^{-1/5} \right\}.
\]

Hence, by (3.16) and (3.17), the first and second 0-terms of (3.9) are

\[
O(\zeta(k)x^{1/r} \exp \left\{ - \frac{A}{2} r^{-8/5} U^{3/5} V^{-1/5} \right\} \log x)
\]

\[
= O(\zeta(r+1)x^{1/r} \exp \left\{ - \frac{A}{2} r^{-8/5} U^{3/5} V^{-1/5} \right\} \log x), \text{ since } k \geq r + 1
\]

\[
= O(x^{1/r} \exp \left\{ - \frac{A}{2} r^{-8/5} U^{3/5} V^{-1/5} \right\} \log x), \text{ uniformly in } k.
\]

Hence, if \( \Delta_{k,r}(x) \) denotes the error term in the asymptotic formula (3.9), then we have

\[
(3.18) \quad \Delta_{k,r}(x) = O(x^{1/r} \exp \left\{ - \frac{A}{2} r^{-8/5} U^{3/5} V^{-1/5} \right\} \log x) + E_{k,r}(x),
\]

where the 0-estimate is uniform in \( k \).

**Case** \( k = 3 \). In this case \( r \) must be \( = 2 \). By (3.6) and (3.17), we have

\[
E_{3,2}(x) = O(x^{1/2} \exp \left\{ - \frac{A}{6} (2)^{-8/5} U^{3/5} V^{-1/5} \right\} \log x),
\]

so that by (3.18),

\[
(3.19) \quad \Delta_{3,2}(x) = O(x^{1/2} \exp \left\{ - B \log^{3/5} x (\log \log x)^{-1/5} \right\}),
\]

where \( B \) is a positive constant \( \left( 0 < B < \frac{A}{6} (2)^{-8/5} \right) \).

**Case** \( k = 4 \). In this case \( r = 2 \) or \( 3 \). Since \( \frac{4}{3} < r < \frac{5}{2} \), we have \( 0 < 1 - rz < 1 \). By (3.6) and (3.17), we have

\[
E_{4,r}(x) = O\left( x^{1/r} \exp \left\{ - \frac{A(1 - rz)}{2} r^{-8/5} U^{3/5} V^{-1/5} \right\} \right).
\]

Again, since \( 0 < 1 - rz < 1 \), the first 0-term in (3.18) is also of the above order of \( E_{4,r}(x) \). Hence

\[
(3.20) \quad \Delta_{4,r}(x) = O(x^{1/r} \exp \left\{ - B \log^{3/5} x (\log \log x)^{-1/5} \right\}),
\]

where \( B \) is a positive constant.
Case $k \geq 5$. In this case $r = 2, 3$ or $4 \leq r < k$. When $r = 2$ or $3$, by (3.6) and (3.17), we have

$$E_{k,r}(x) = O\left(x^{1/r} \exp \left\{ -\frac{A(1 - rz)}{2} r^{-8/5} U^{3/5} V^{-1/5} \right\} \right),$$

so that by (3.18),

$$\Delta_{k,r}(x) = O(x^{1/r} \exp \{ -B \log^{3/5} x (\log \log x)^{-1/5} \}),$$

where $B$ is a positive constant and the $O$-estimate is uniform in $k$.

When $4 \leq r < k$, by (3.6), $E_{k,r}(x) = O(x^{a})$ and the first $O$-term in (3.18) is $O(x^{1/r})$, so that we have

$$\Delta_{k,r}(x) = O(x^{a}),$$

where the $O$-estimate is uniform in $k$.

Hence, by (3.9), (3.18)–(3.22), Theorem 1 follows.

4. Proof of theorem 2

Following the same procedure adopted in the proof of theorem 1 and making use of (2.10) and (2.11) instead of (2.6) and (2.7) we get that

$$\Delta_{k,r}(x) = O\left(\rho^{1/2 - rz/2} \omega(\rho z) \log z \right) + O\left(\rho^{1/2 - rz/2} \omega(\rho z) \log \left(\frac{1}{\rho}\right) \right) + E_{k,r}(x),$$

where the $O$-estimates are uniform in $k$ and $E_{k,r}(x)$ is given by (3.6).

Case $k = 3$. In this case $r$ must be $2$. Choosing $\rho = z^{-3/11}$, we see that $0 < \rho < 1$, $\frac{1}{\rho} < z$, so that $\log \left(\frac{1}{\rho}\right) < \log z$, and

$$\rho^{1/2 - rz/2} z^{1/2} = \rho^{1/2} z = x^{5/11}. $$

Since $\omega(x)$ is monotonic increasing, $\omega(\rho z) < \omega(z)$. Hence, by (4.1), (3.6) and the above, we have

$$\Delta_{3,2}(x) = O(x^{5/11} \omega(x^{1/2}) \log x) + O(x^{5/11} \log x) = O(x^{5/11} \omega(x)).$$

Case $k = 4$. In this case $r = 2$ or $3$. Choosing $\rho = z^{-1/(1 + 2r(1 - s))}$, we see that $0 < \rho < 1$, $\frac{1}{\rho} < z$, so that $\log \left(\frac{1}{\rho}\right) < \log z$, and

$$\rho^{1/2 - rz/2} = \rho^{1 - rz} z = x^{2-s/(1+2r(1-s)).}$$

Since $\omega(x)$ is monotonic increasing, $\omega(\rho z) < \omega(z)$. Hence by (4.1), (3.6) and the
above, we have
\[(4.3)\]
\[\Delta_k(x) = O(x^{2-a(1+2r(1-a))} \omega(x^{1/2}) \log x) = O(x^{2-a(1+2r(1-a))} \omega(x)).\]

Case \(k \geq 5\). In this case \(r = 2, 3\) or \(4 \leq r < k\). When \(r = 2\) or \(3\), we have by (3.6), \(E_{k,r}(x) = O(\rho^{1-r} x)\). Choosing \(\rho = z^{-((1+2r(1-a))} \), as in the case \(k = 4\), we get that
\[(4.4)\]
\[\Delta_{k,r}(x) = O(x^{(2-a)/(1+(2r(1-a))} \omega(x)),\]
where the \(O\)-estimate is uniform in \(k\). When \(4 \leq r < k\), by (3.6), we have \(E_{k,r}(x) = O(x^a)\). We have \(\omega(x) = O(x^a)\) and \(\log z = O(x^a)\) for every \(\varepsilon > 0\). We assume that \(0 < \varepsilon < 1\). Hence, by (4.1), we have
\[(4.5)\]
\[\Delta_{k,r}(x) = O(\rho^{1/2-r+\varepsilon} z^{1/2+2t}) + O\left(\rho^{1/2-r+\varepsilon} z^{1/2+\varepsilon} \log \left(\frac{1}{\rho}\right)\right) + O(x^a).\]

Now, choosing \(\rho = z^{-(2a-1+4\varepsilon)/(2r-1-2\varepsilon)}\), we see that \(0 < \rho < 1\), \(\frac{1}{\rho} < z\), so that
\[\log \left(\frac{1}{\rho}\right) < \log z = O(x^a)\] and
\[\rho^{1/2-r+\varepsilon} z^{1/2+2t} = x^a.\]

Hence, by (4.5), we have
\[(4.6)\]
\[\Delta_{k,r}(x) = O(x^a),\]
where the \(O\)-estimate is uniform in \(k\). Hence, by (4.2), (4.3), (4.4) and (4.6), Theorem 2 follows.

**Remark.** In the case \(4 \leq r < k\), we may choose the function \(\rho = \rho(x)\), which tends to zero as \(x \to \infty\) to be a function which tends to zero more rapidly than that chosen above. In such a case, although the first and second \(O\)-terms in (4.5) are \(O(x^a)\), where \(\beta < \alpha\), but because of the third \(O\)-term in (4.5), we again get \(\Delta_{k,r}(x) = O(x^a)\). Hence we can not improve the result that \(\Delta_{k,r}(x) = O(x^a)\) for \(4 \leq r < k\), even on the assumption of the Riemann hypothesis.

**References**


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