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THE DIVISOR PROBLEM FOR (k, r)-INTEGERS

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THE DIVISOR PROBLEM FOR (k, r) — INTEGERS[1]

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1. Introduction

Let k and r be fixed integers such that $1 < r < k$. It is well-known that a positive integer is called r-free if it is not divisible by the r-th power of any integer > 1. We call a positive integer n, a (k,r)-integer, if n is of the form $n = a^k b$, where a is a positive integer and b is a r-free integer. In the limiting case, when k becomes infinite, a (k,r)-integer becomes a r-free integer and so one might consider the (k,r) integers as generalized r-free integers.

It has been shown by one of the authors and V. Siva Rama Prasad [4] that if $\tau_{(r)}(n)$ denotes the number of r-free divisors of n, then for $x \geqq 3$,

$$\sum_{n \leqq x} \tau_{(r)}(n) = \frac{x}{\zeta(r)}\left(\log x + 2\gamma - 1 - \frac{r\zeta'(r)}{\zeta(r)}\right) + \Delta_r(x), \tag{1.1}$$

where $\Delta_r(x) = O(x^{1/r}\delta(x))$ or $O(x^{\alpha})$, according as $r = 2, 3$ or $r \geqq 4$; $\delta(x) = \exp\{-A\log^{3/5}x(\log\log x)^{-1/5}\}$, A being a positive constant and α is the number which appears in the Dirichlet divisor problem

$$\sum_{n \leqq x} \tau(n) = x(\log x + 2\gamma - 1) + O(x^{\alpha}), \tag{1.2}$$

where $\tau(n)$ is the number of divisors of n.

It is known that $\frac{1}{4} < \alpha < \frac{1}{3}$ (cf. [1], p. 272). The best result yet proved has been obtained recently by Kolesnik [2], who proved that the error term in (1.2) is $O(x^{(12/37)+\varepsilon})$, for any $\varepsilon > 0$. There is a conjecture that $\alpha = \frac{1}{4} + \varepsilon$. In the formula (1.1), $\zeta(s)$, denotes the Riemann Zeta function and $\zeta'(s)$ its derivative and γ is Euler's constant.

It has also been shown in [4] on the assumption of the Riemann hypothesis that $\Delta_2(x) = O(x^{(2-\alpha)/(5-4\alpha)}\omega(x))$, $\Delta_3(x) = O(x^{(2-\alpha)/(7-6\alpha)}\omega(x))$ and $\Delta_r(x) = O(x^{\alpha})$

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for $r \geqq 4$, where $\omega(x) = \exp\{A \log x (\log\log x)^{-1}\}$, A being a positive constant. For earlier (weaker) estimations of $\Delta_r(x)$ by various authors, we refer to the bibliography given in [4].

Let us call a divisor d of a positive integer n, *a (k,r)-divisor of n* if d is a (k,r)-integer. Let $\tau_{(k,r)}(n)$ denote the number of (k,r)-divisors of n. The object of this paper is to prove the following:

THEOREM 1. *For $1 < r < k$ and $x \geqq 3$,*

$$\sum_{n \leqq x} \tau_{(k,r)}(n) = \frac{\zeta(k)x}{\zeta(r)}\left(\log x + 2\gamma - 1 - \frac{r\zeta'(r)}{\zeta(r)} + \frac{k\zeta'(k)}{\zeta(k)}\right) + \Delta_{k,r}(x), \tag{1.3}$$

where $\Delta_{k,r}(x) = O(x^{1/r}\delta(x))$ or $O(x^\alpha)$, according as $r = 2,3$ or $4 \leqq r < k$, the 0-estimates being uniform in k; $\delta(x) = \exp\{-B\log^{3/5}x(\log\log x)^{-1/5}\}$, B being a positive constant and α is the number which appears in (1.2).

THEOREM 2. *If the Riemann hypothesis is true, then the error term $\Delta_{k,r}(x)$ in* (1.3) *has the following improved* 0*-estimates*:

$$\Delta_{3,2}(x) = O(x^{5/11}\omega(x)),\ \Delta_{k,2}(x) = O(x^{(2-\alpha)/(5-4\alpha)}\omega(x))$$

for $k \geqq 4$, $\Delta_{k,3}(x) = O(x^{(2-\alpha)/(7-6\alpha)}\omega(x))$ for $k \geqq 4$ and $\Delta_{k,r}(x) = O(x^\alpha)$ for $4 \leqq r < k$; where the 0-estimates are uniform in k and $\omega(x) = \exp\{A\log x (\log\log x)^{-1}\}$, A being a positive constant and α is given by (1.2).

It may be noted that in the limiting case when $k \to \infty$, formula (1.3) coincides with (1.1) and the 0-estimates of $\Delta_r(x) = \Delta_{\infty,r}(x)$ obtained in [4] follow as a particular case.

2. Prerequisites

In this section we prove some lemmas which are needed in the proofs of Theorem 1 and 2. Throughout the following, x denotes a real variable $\geqq 3$. The following elementary estimates are well-known:

$$\sum_{n \leqq x} \frac{1}{n^s} = O(x^{1-s}) \text{ if } 0 \leqq s < 1. \tag{2.1}$$

$$\sum_{n > x} \frac{1}{n^s} = \zeta(s) - \sum_{n \leqq x} \frac{1}{n^s} = 0\left(\frac{1}{x^{s-1}}\right) \text{ if } s > 1. \tag{2.2}$$

$$\sum_{n < x} \frac{\log n}{n^s} = -\zeta'(s) - \sum_{n \leqq x} \frac{\log n}{n^s} = 0\left(\frac{\log x}{x^{s-1}}\right) \text{ if } s > 1. \tag{2.3}$$

LEMMA 2.1 (*cf*,. [6]; *Satz* 3, p. 191).

$$M(x) = \sum_{n \leqq x} \mu(n) = O(x\delta(x)), \tag{2.4}$$

where

$$\delta(x) = \exp\{-A\log^{3/5}x(\log\log x)^{-1/5}\}, \tag{2.5}$$

A being a positive constant.

LEMMA 2.2 (*cf.* [4] *Lemma* 2.2). *For any* $s > 1$,

$$\sum_{n\leq x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O\left(\frac{\delta(x)}{x^{s-1}}\right). \tag{2.6}$$

LEMMA 2.3 (*cf.* [4], *Lemma* 2.3). *For any* $s > 1$,

$$\sum_{n\leq x} \frac{\mu(n)\log n}{n^s} = \frac{\zeta'(s)}{\zeta^2(s)} + O\left(\frac{\delta(x)\log x}{x^{s-1}}\right). \tag{2.7}$$

LEMMA 2.4 (*cf.* [5], *Theorem* 14-26 (*A*), p. 316). *If the Riemann hypothesis is true, then*

$$M(x) = \sum_{n\leq x} \mu(n) = O(x^{1/2}\omega(x)), \tag{2.8}$$

where

$$\omega(x) = \exp\{A\log x(\log\log x)^{-1}\}, \tag{2.9}$$

A being a positive constant.

LEMMA 2.5 (*cf.* [4], *Lemma* 2.5). *If the Riemann hypothesis is true, then for any* $s > 1$,

$$\sum_{n\leq x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O(x^{\frac{1}{2}-s}\,\omega(x)). \tag{2.10}$$

LEMMA 2.6 (*cf.* [4], *Lemma* 2.6). *If the Riemann hypothesis is true, then for any* $s > 1$,

$$\sum_{n\leq x} \frac{\mu(n)\log n}{n^s} = \frac{\zeta'(s)}{\zeta^2(s)} + O(x^{\frac{1}{2}-s}\,\omega(x)\log x). \tag{2.11}$$

LEMMA 2.7 (*cf.* [3], *Lemma* 2.6). *If* $q_{k,r}(n)$ *denotes the characteristic function of the set of* (k,r)*-integers, that is,* $q_{k,r}(n) = 1$ *or* 0 *according as* n *is or is not a* (k,r)*-integer, then*

$$q_{k,r}(n) = \sum_{a^kb^rc=n} \mu(b). \tag{2.12}$$

LEMMA 2.8. $\tau_{(k,r)}(n) = \sum_{a^kb^rc=n}\mu(b)\tau(c)$.

PROOF. We have $\tau_{(k,r)}(n) = \sum_{d\delta=n} q_{k,r}(d)$, so that by (2.12),

$$\tau_{(k,r)}(n) = \sum_{d\delta=n}\ \sum_{a^kb^rc=d} \mu(b) = \sum_{a^kb^rc\delta=n} \mu(b)$$

$$= \sum_{a^k b^r | n} \mu(b) \sum_{c\delta = (n/a^k b^r)} 1 = \sum_{a^k b^r | n} \mu(b)\tau\left(\frac{n}{a^k b^r}\right)$$

$$= \sum_{a^k b^r c = n} \mu(b)\tau(c).$$

Hence Lemma 2.8 follows.

LEMMA 2.9. *For* $k \geqq 3$,

$$\sum_{a^k c \leqq x} \tau(c) = \zeta(k)x\left(\log x + 2\gamma - 1 + \frac{k\zeta'(k)}{\zeta(k)}\right) + R_k(x), \tag{2.13}$$

where

(2.14) $R_k(x) = O(x^{\frac{1}{3}} \log x)$ or $O(x^{\alpha})$, *according as* $k = 3$ *or* $k \geqq 4$, *where the second O-estimate is uniform in* k

PROOF. We have by (1.2), (2.2) and (2.3),

$$\begin{aligned}
\sum_{a^k c \leqq x} \tau(c) &= \sum_{a \leqq \sqrt[k]{x}} \sum_{c \leqq x/a^k} \tau(c) \\
&= \sum_{a \leqq \sqrt[k]{x}} \left\{\frac{x}{a^k}\left(\log\frac{x}{a^k} + 2\gamma - 1\right) + O\left(\frac{x^{\alpha}}{a^{k\alpha}}\right)\right\} \\
&= x(\log x + 2\gamma - 1)\sum_{a \leqq \sqrt[k]{x}} \frac{1}{a^k} - kx \sum_{a \leqq \sqrt[k]{x}} \frac{\log a}{a^k} + O\left(x^{\alpha} \sum_{a \leqq \sqrt[k]{x}} a^{-k\alpha}\right) \\
&= x(\log x + 2\gamma - 1)\{\zeta(k) + O(x^{-1+(1/k)})\} - kx\{-\zeta'(k) \\
&\qquad + O\left(\frac{\log x}{x^{1-1/k}}\right)\} + O\left(x^{\alpha} \sum_{a \leqq \sqrt[k]{x}} a^{-k\alpha}\right) \\
&= \zeta(k)x\left(\log x + 2\gamma - 1 + \frac{k\zeta'(k)}{\zeta(k)}\right) + O(x^{1/k}\log x) + O\left(x^{\alpha} \sum_{a \leqq \sqrt[k]{x}} a^{-k\alpha}\right).
\end{aligned}$$

Since $\frac{1}{4} < \alpha < \frac{1}{3}$, we have $k\alpha \lesseqgtr 1$ according as $k = 3$ or $k \geqq 4$. Hence, by (2.1) and (2.2), the last O-term in the above is $0(x^{\frac{1}{3}})$ or $O(\zeta(k\alpha)x^{\alpha}) = O(\zeta(4\alpha)x^{\alpha}) = O(x^{\alpha})$, uniformly in k, according as $k = 3$ or $k \geqq 4$. Hence Lemma 2.9 follows.

3. Proof of Theorem 1

By Lemma 2.8, we have

$$\tau_{(k\ r)}(n) = \sum_{a^k b^r c = n} \mu(b)\tau(c).$$

Hence

$$(3.1)\qquad \sum_{n\le x}\tau_{(k,r)}(n)=\sum_{n\le x}\sum_{a^kb^rc=n}\mu(b)\tau(c)=\sum_{a^kb^rc\le x}\mu(b)\tau(c),$$

where the summation on the right being taken over all ordered triads (a,b,c) such that $a^kb^rc\le x$.

Let $z=x^{1/r}$. Further, let $0<\rho=\rho(x)<1$, where the function $\rho(x)$ will be suitably chosen later.

Now, if $a^kb^rc\le x$, then both $b>\rho z$ and $a^kc>\rho^{-r}$ can not simultaneously hold good. Hence from (3.1), we have

$$(3.2)\quad \sum_{n\le x}\tau_{(k,r)}(n)=\sum_{\substack{a^kb^rc\le x\\ b\le\rho z}}\mu(b)\tau(c)+\sum_{\substack{a^kb^rc\le x\\ a^kc\le\rho^{-r}}}\mu(b)\tau(c)-\sum_{\substack{b\le\rho z\\ a^kc\le\rho^{-r}}}\mu(b)\tau(c)$$
$$=S_1+S_2-S_3,\ \text{say}.$$

By (2.13), we have

$$(3.3)\quad S_1=\sum_{\substack{a^kb^rc\le x\\ b\le\rho z}}\mu(b)\tau(c)=\sum_{b\le\rho z}\mu(b)\sum_{a^kc\le(x/b^r)}\tau(c)$$
$$=\sum_{b\le\rho z}\mu(b)\left\{\zeta(k)\frac{x}{b^r}\left(\log\frac{x}{b^r}+2\gamma-1+\frac{k\zeta'(k)}{\zeta(k)}\right)+R_k\left(\frac{x}{b^r}\right)\right\}$$
$$=\zeta(k)x\left(\log x+2\gamma-1+\frac{k\zeta'(k)}{\zeta(k)}\right)\sum_{b\le\rho z}\frac{\mu(b)}{b^r}$$
$$-\zeta(k)rx\sum_{b\le\rho z}\frac{\mu(b)\log b}{b^r}+E_{k,r}(x),$$

where

$$(3.4)\qquad E_{k,r}(x)=\sum_{b\le\rho z}\mu(b)R_k\left(\frac{x}{b^r}\right).$$

Hence by (3.3), (2.6) and (2.7), we have

$$(3.5)\quad S_1=\zeta(k)x\left(\log x+2\gamma-1+\frac{k\zeta'(k)}{\zeta(k)}\right)\left\{\frac{1}{\zeta(r)}+O\left(\frac{\delta(\rho z)}{(\rho z)^{r-1}}\right)\right\}$$
$$-\zeta(k)rx\left\{\frac{\zeta'(r)}{\zeta(r)}+O\left(\frac{\delta(\rho z)\log(\rho z)}{(\rho z)^{r-1}}\right)\right\}+E_{k,r}(x)$$
$$=\frac{\zeta(k)x}{\zeta(r)}\left(\log x+2\gamma-1-\frac{r\zeta'(r)}{\zeta(r)}+\frac{k\zeta'(k)}{\zeta(k)}\right)$$
$$+O(\zeta(k)\rho^{1-r}z\delta(\rho z)\log z)+E_{k,r}(x).$$

By (2.14) and (3.4), we have

$$E_{k,r}(x)=O\left(\sum_{b\le\rho z}\frac{x^{\frac{1}{3}}}{b^{r/3}}\log\left(\frac{x}{b^r}\right)\right)\text{ or }O\left(\sum_{b\le\rho z}\frac{x^{\alpha}}{b^{r/\alpha}}\right),$$

according as $k = 3$ or $k \geqq 4$. Since $1 < r < k$, we have $r = 2$, when $k = 3$ and since $\frac{1}{4} < \alpha < \frac{1}{3}$, we have by (2.1) and (2.2), the following 0-estimates:

$$\text{(3.6)} \qquad \begin{cases} E_{3,2}(x) = O(\rho^{1/3}\, x^{1/2} \log x) \\ E_{4,r}(x) = O(\rho^{1-r\alpha} z) \\ E_{k,r}(x) = O(\rho^{1-r\alpha} z) \text{ or } O(x^{\alpha}), \\ \text{according as } r = 2,3 \text{ or } 4 \leqq r < k; \end{cases}$$

where the 0-estimates are uniform in k. We have

$$\begin{aligned} S_2 &= \sum_{\substack{a^k b^r c \leqq x \\ a^k c \leqq \rho^{-r}}} \mu(b)\tau(c) = \sum_{a^k c \leqq \rho^{-r}} \tau(c) \sum_{b \leqq \sqrt[r]{(x/a^k c)}} \mu(b) \\ &= \sum_{a^k c \leqq \rho^{-r}} \tau(c) M\left(\sqrt[r]{\frac{x}{a^k c}}\right) \\ &= 0\left(x^{1/r} \sum_{a^k c \leqq \rho^{-r}} \tau(c) a^{-k/r} c^{-1/r} \delta\left(\sqrt[r]{\frac{x}{a^k c}}\right)\right), \end{aligned}$$

by (2.4). Since $\delta(x)$ is monotonic decreasing and $\sqrt[r]{\dfrac{x}{a^k c}} \geqq \delta z$, we have $\delta\left(\sqrt[r]{\dfrac{x}{a^k c}}\right) \leqq \delta(\rho z)$. Also, by (2.1), (2.2) and (1.2),

$$\begin{aligned} \sum_{a^k c \leqq \rho^{-r}} \tau(c) a^{-k/r} c^{-1/r} &= \sum_{a \leqq \rho^{-r/k}} a^{-k/r} \sum_{c \leqq \rho^{-r} a^{-k}} \tau(c) c^{-1/r} \\ &= O\left(\sum_{a \leqq \rho^{-r/k}} a^{-k/r} (\rho^{-r} a^{-k})^{1-(1/r)} \log(\rho^{-r} a^{-k})\right) \\ &= O\left(\rho^{1-r} \log\left(\frac{1}{\rho}\right) \sum_{a \leqq \rho^{-r/k}} a^{-k}\right) \\ &= O\left(\zeta(k) \rho^{1-r} \log\left(\frac{1}{\rho}\right)\right). \end{aligned}$$

Hence

$$\text{(3.7)} \qquad S_2 = O\left(\zeta(k)\rho^{1-r} z \delta(\rho z (\log\left(\frac{1}{\rho}\right)\right).$$

Further, we have by (2.4) and (2.13),

$$\text{(3.8)} \qquad \begin{aligned} S_3 &= \sum_{\substack{b \leqq \rho z \\ a^k c \leqq \rho^{-r}}} \mu(b)\tau(c) = \sum_{b \leqq \rho z} \mu(b) \sum_{a^k c \leqq \rho^{-r}} \tau(c) \\ &= M(\rho z) \sum_{a^k c \leqq \rho^{-r}} \tau(c) \\ &= O(\rho z \delta(\rho z) \zeta(k) \rho^{-r} \log(\rho^{-r})) \end{aligned}$$

$$= O\left(\zeta(k)\rho^{1-r}z\delta(\rho z)\log\left(\frac{1}{\rho}\right)\right).$$

Hence by (3.2), (3.5), (3.7) and (3.8)

$$\sum_{n\leqq x}\tau_{(k,r)}(n) = \frac{\zeta(k)x}{\zeta(r)}\left(\log x + 2\gamma - 1 - \frac{r\zeta'(r)}{\zeta(r)} + \frac{k\zeta'(k)}{\zeta(k)}\right)$$

$$(3.9) \qquad + 0(\zeta(k)\rho^{1-r}z\delta(\rho z)\log z)$$

$$+ 0\left(\zeta(k)\rho^{1-r}z\delta(\rho z)\log\left(\frac{1}{p}\right)\right) + E_{k,r}(x).$$

Now, we choose,

$$(3.10) \qquad \rho = \rho(x) = \{\delta(x^{1/2r})\}^{1/r},$$

and write

$$(3.11) \qquad f(x) = \log^{3/5}(x^{1/2r})\{\log\log(x^{1/2r})\}^{-1/5}$$

$$= \left(\frac{1}{2r}\right)^{3/5} U^{3/5}(V - \log 2r)^{-1/5},$$

where $U = \log x$ and $V = \log\log x$.

(3.12) For $V \geqq 2\log 2r$, that is, $U \geqq 4r^2$, $x \geqq \exp(4r^2)$, we have

$$V^{-1/5} \leqq (V - \log 2r)^{-1/5} \leqq \left(\frac{V}{2}\right)^{-1/5}$$

and therefore

$$(3.13) \qquad \tfrac{1}{2}r^{-3/5}U^{3/5}V^{-1/5} \leqq f(x) \leqq r^{-3/5}U^{3/5}V^{-1/5}.$$

(3.14) We assume without loss of generality that the constant A in (2.5) is less than 1.

By (3.10), (2.5) and (3.11), we have

$$(3.15) \qquad \rho = \exp\ -\frac{A}{r}f(x)\}.$$

By (3.12), we have

$$r^{-8/5}U^{3/5}V^{-1/5} \leqq \frac{U}{2r}.$$

Hence, by (3.13), (3.14), (3.15) and the above,

$$\rho \geqq \exp(-A\,r^{-8/5}U^{3/5}V^{-1/5}) \geqq \exp(-r^{-8/5}U^{3/5}V^{-1/5})$$

$$\geqq \exp\left(-\frac{U}{2r}\right) = \exp\left(-\frac{\log x}{2r}\right),$$

so that $\rho \geqq x^{-(1/2r)}$.

(3.16) $\log\left(\frac{1}{\rho}\right) \leqq \log(\sqrt{z}) = 0(\log x)$ and $\rho z \geqq x^{1/(2r)}$.

Since $\delta(x)$ is monotonic decreasing, we have $\delta(\rho z) \leqq \delta(x^{1/(2r)}) = \rho^r$, by (3.10), so that by (3.13) and (3.15), we have

$$\rho^{1-r}\delta(\rho z) \leqq \rho \leqq \exp\left\{-\frac{A}{2}r^{-8/5}U^{3/5}V^{-1/5}\right\}. \tag{3.17}$$

Hence, by (3.16) and (3.17), the first and second 0-terms of (3.9) are

$$O(\zeta(k)x^{1/r}\exp\{-\frac{A}{2}r^{-8/5}U^{3/5}V^{-1/5}\}\log x)$$
$$= O(\zeta(r+1)x^{1/r}\exp\{-\frac{A}{2}r^{-8/5}U^{3/5}V^{-1/5}\}\log x), \text{ since } k \geqq r+1$$
$$= O(x^{1/r}\exp\{-\frac{A}{2}r^{-8/5}U^{3/5}V^{-1/5}\}\log x, \text{ uniformly in } k.$$

Hence, if $\Delta_{k,r}(x)$ denotes the error term in the asymptotic formula (3.9), then we have

$$\Delta_{k,r}(x) = O(x^{1/r}\exp\{-\frac{A}{2}r^{-8/5}U^{3/5}V^{-1/5}\}\log x) + E_{k,r}(x), \tag{3.18}$$

where the 0-estimate is uniform in k.

Case $k = 3$. In this case r must be $= 2$. By (3.6) and (3.17), we have

$$E_{3,2}(x) = O(x^{1/2}\exp\{-\frac{A}{6}(2)^{-8/5}U^{3/5}V^{-1/5}\}\log x),$$

so that by (3.18),

$$\Delta_{3,2}(x) = O(x^{1/2}\exp\{-B\log^{3/5}x(\log\log x)^{-1/5}\}), \tag{3.19}$$

where B is a positive constant $\left(0 < B < \frac{A}{6}(2)^{-8/5}.\right)$

Case $k = 4$. In this case $r = 2$ or 3. Since $\frac{1}{4} < \alpha < \frac{1}{3}$, we have $0 < 1 - r\alpha < 1$. By (3.6) and (3.17), we have

$$E_{4,r}(x) = O\left(x^{1/r}\exp\left\{-\frac{A(1-r\alpha)}{2}r^{-8/5}U^{3/5}V^{-1/5}\right\}\right).$$

Again, since $0 < 1 - r\alpha < 1$, the first 0-term in (3.18) is also of the above order of $E_{4,r}(x)$. Hence

$$\Delta_{4,r}(x) = O(x^{1/r}\exp\{-B\log^{3/5}x(\log\log x)^{-1/5}\}), \tag{3.20}$$

where B is a positive constant.

Case $k \geqq 5$. In this case $r = 2,3$ or $4 \leqq r < k$. When $r = 2$ or 3, by (3.6) and (3.17), we have

$$E_{k,r}(x) = O\left(x^{1/r}\exp\left\{-\frac{A(1-r\alpha)}{2}\,r^{-8/5}U^{3/5}V^{-1/5}\right\}\right),$$

so that by (3.18),

$$\Delta_{k,r}(x) = O(x^{1/r}\exp\{-B\log^{3/5}x(\log\log x)^{-1/5}\}), \tag{3.21}$$

where B is a positive constant and the 0-estimate is uniform in k.

When $4 \leqq r < k$, by (3.6), $E_{k,r}(x) = O(x^{\alpha})$ and the first O-term in (3.18) is $O(x^{1/r})$, so that we have

$$\Delta_{k,r}(x) = O(x^{\alpha}), \tag{3.22}$$

where the 0-estimate is uniform in k.

Hence, by (3.9), (3.18)-(3.22), Theorem 1 follows.

4. Proof of theorem 2

Following the same procedure adopted in the proof of theorem 1 and making use of (2.10) and (2.11) instead of (2.6) and (2.7) we get that

$$\Delta_{k,r}(x) = O\left(\rho^{1/2-r}z^{1/2}\omega(\rho z)\log z\right) + O\left(\rho^{1/2-r}z^{1/2}\omega(\rho z)\log\left(\frac{1}{\rho}\right)\right) + E_{k,r}(x), \tag{4.1}$$

where the 0-estimates are uniform in k and $E_{k,r}(x)$ is given by (3.6).

Case $k = 3$. In this case r must be $= 2$. Choosing $\rho = z^{-3/11}$, we see that $0 < \rho < 1, \frac{1}{\rho} < z$, so that $\log\left(\frac{1}{\rho}\right) < \log z$, and

$$\rho^{1/2-2}z^{1/2} = \rho^{1/3}z = x^{5/11}.$$

Since $\omega(x)$ is monotonic increasing, $\omega(\rho z) < \omega(z)$. Hence, by (4.1), (3.6) and the above, we have

$$\begin{aligned}\Delta_{3,2}(x) &= O(x^{5/11}\omega(x^{1/2})\log x) + O(x^{5/11}\log x) \\ &= O(x^{5/11}\omega(x)).\end{aligned} \tag{4.2}$$

Case $k = 4$. In this case $r = 2$ or 3. Choosing $\rho = z^{-1/(1+2r(1-\alpha))}$, we see that $0 < \rho < 1, \frac{1}{\rho} < z$, so that $\log\left(\frac{1}{\rho}\right) < \log z$, and

$$\rho^{1/2-r}z^{1/2} = \rho^{1-r\alpha}z = x^{2-\alpha/(1+2r(1-\alpha))}.$$

Since $\omega(x)$ is monotonic increasing, $\omega(\rho z) < \omega(z)$. Hence by (4.1), (3.6) and the above, we have

(4.3) $$\Delta_{4,r}(x) = O(x^{2-\alpha/(1+2r(1-\alpha)}\omega(x^{1/2})\log x)$$

$$= O(x^{2-\alpha/(1+2r(1-\alpha)}\omega(x)).$$

Case $k \geqq 5$. In this case $r = 2,3$ or $4 \leqq r < k$. When $r = 2$ or 3, we have by (3.6), $E_{k,r}(x) = O(\rho^{1-r\alpha} z)$. Choosing $\rho = z^{-(1/(1+2r(1-\alpha))}$, as in the case $k = 4$, we get that

(4.4) $$\Delta_{k,r}(x) = O(x^{(2-\alpha)/(1+(2r(1-\alpha))}\omega(x)),$$

where the O-estimate is uniform in k. When $4 \leqq r < k$, by (3.6), we have $E_{k,r}(x) = O(x^{\alpha})$. We have $\omega(x) = O(x^{\varepsilon})$ and $\log z = O(x^{\varepsilon})$ for every $\varepsilon > 0$. We assume that $0 < \in < 1$. Hence, by (4.1), we have

(4.5) $$\Delta_{k,r}(x) = O(\rho^{1/2-r+\varepsilon} z^{1/2+2\varepsilon}) + O\left(\rho^{1/2-r+\varepsilon} z^{1/2+\varepsilon}\log\left(\frac{1}{\rho}\right)\right) + O(x^{\alpha}).$$

Now, choosing $\rho = z^{-(2r\alpha-1+4\varepsilon)/(2r-1-2\varepsilon)}$, we see that $0 < \rho < 1$, $\dfrac{1}{\rho} < z$, so that $\log\left(\dfrac{1}{\rho}\right) < \log z = O(z^{\varepsilon})$ and

$$\rho^{1/2-r+\varepsilon} z^{1/2+2\varepsilon} = x^{\alpha}.$$

Hence, by (4.5), we have

(4.6) $$\Delta_{k,r}(x) = O(x^{\alpha}),$$

where the O-estimate is uniform in k. Hence, by (4.2), (4.3), (4.4) and (4.6), Theorem 2 follows.

REMARK. In the case $4 \leqq r < k$, we may choose the function $\rho = \rho(x)$, which tends to zero as $x \to \infty$ to be a function which tends to zero more rapidly than that chosen above. In such a case, although the first and second O-terms in (4.5) are $0(x^{\beta})$, where $\beta < \alpha$, but because of the third 0-term in (4.5), we again get $\Delta_{k,r}(x) = O(x^{\alpha})$. Hence we can not improve the result that $\Delta_{k,r}(x) = O(x^{\alpha})$ for $4 \leqq r < k$, even on the assumption of the Riemann hypothesis.

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