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SOME THEOREMS IN ADDITIVE NUMBER THEORY[1]

By

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1. Introduction

In 1922, G. H. HARDY and J. E. LITTLEWOOD [4] in paper III of their celebrated series of papers entitled "Partitio Numerorum" made a number of conjectures. Two of these (conjectures H and L, p. 609 and p. 611 of [4]) are as follows: (a) Every large number is either a square or the sum of a prime and a square (b) Every large number is either a cube or the sum of a prime and a (positive) cube. In each of these conjectures, they have stated the asymptotic expressions also for the number of representations.

The above two conjectures are still open and are made on the basis of the extended Riemann Hypothesis (e.R.H.), namely that every zero of every Dirichlet's function

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{x_q(m)}{m^s},$$

where $\chi_q(m)$ is a character (mod q) has a real part which does not exceed $1/2$ for all q and all χ_q. However, it has been shown by C. HOOLEY [5] in 1957 that every large integer is the sum of a prime and two squares (cf. [4], conjecture J, p. 610) assuming the e.R.H. and by JU. V. LINNIK (cf. [7], also cf. [8], ch. VII) in 1960 without any hypothesis. In 1929, it has been shown by G. K. STANELY (cf. [13], Theorem G) that every large integer is the sum of two primes and a square assuming the e.R.H. and by T. ESTERMANN [2] in 1937 without assuming any hypothesis. In 1968, R. J. MICH [9] has shown that "nearly every" integer n is expressible in the form $n = p + m^2$. For a precise statement of his result we refer to [9].

The problems corresponding to (a) and (b) above involving 4th or higher powers have not received much attention. The utmost that is known in this direction is the result of L. K. HUA (cf. [6], p. 179), namely that every large integer can be expressed as the sum of a prime and s k-th

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powers of integers if $s \geq s_0 \sim (3k \log k)/2$. In this connection, we might also mention the following result proved quite recently by K. PRACHAR [12]: Given a positive integer, l, there exist constants $\eta > 1$ and $\delta > 0$ such that for large N at least δN of the positive integers up to N are not expressible in the form $p + m^l$, where p is a prime and m is a positive integer not exceeding $\eta \log N$.

In a different direction, T. ESTERMANN [1] established in 1931 that every sufficiently large integer n can be expressed as the sum of a prime and a square-free integer and that the number $T(n)$ of such representations is given asymptotically by

$$T(n) \sim \prod_{\substack{p \\ p \nmid n}} \left\{1 - \frac{1}{p(p-1)}\right\} \operatorname{Li} n,$$

where the product is extended over all primes p which do not divide n and

$$\operatorname{Li} n = \int_2^n \frac{\mathrm{d}t}{\log t}.$$

In 1935, A. PAGE (cf. [11], theorem III) established an asymptotic formula for $T(n)$ with an O-estimate for the error term, namely

$$T(n) = \prod_{\substack{p \\ p \nmid n}} \left\{1 - \frac{1}{p(p-1)}\right\} \operatorname{Li} n + O\left(\frac{n}{\log^5 n} (\log \log n)^8 \log \log \log n\right).$$

This O-estimate has been further improved in 1936 by A. WALFISZ [17] to $O\left(\frac{n}{\log^H n}\right)$, where $H > 0$.

Let r be any integer > 1. It is trivial after ESTERMANN's [1] result that every sufficiently large integer can be expressed as the sum of a prime and a r-free integer (that is, a positive integer which is not divisible by the r-th power of any integer > 1), since every square-free integer is a r-free integer. In 1949, L. MIRSKY [10] obtained the following asymptotic formula for the number $T(r; n)$ of representations of n as the sum of a prime and a r-free integer:

$$T(r; n) = \prod_{\substack{p \\ p \nmid n}} \left\{1 - \frac{1}{p^{r-1}(p-1)}\right\} \operatorname{Li} n + O\left(\frac{n}{\log^H n}\right), \tag{1.1}$$

where H is any positive number and the O-estimate depends at most on r and H.

In this paper, we are concerned with the following problem: Let k and r be integers such that $1 < r < k$; and let us define an integer n to be a

(k, r)- integer, if it is of the form $n = a^k b$, where a is a positive integer and b is a r-free integer. Clearly, a (k, r)-integer is a cross between a k-th power of a positive integer and a r-free integer; and the set $Q_{k,r}$ of all (k, r)-integers includes the set S_k of all k-th power integers and the set Q_r of all r-free integers. Since every large integer is the sum of a prime and a r-free integer, it is trivial, that every large integer is the sum of a prime and a (k, r)-integer. However, let us define a proper (k, r)-integer as a (k, r)-integer which is not a r-free integer; these are integers of the form $n = a^k b$, where $a > 1$ and b is a r-free integer. The set $Q^*_{k,r}$ of such integers includes the set S_k, but excludes the set Q_r. The problem with which we are concerned in this paper is as follows: Is every sufficiently large integer n be expressible as the sum of a prime and a proper (k, r)-integer.

We answer this problem in the affirmative. In fact, we obtain an asymptotic formula for the number $T^*(k, r; n)$ of representations of n as the sum of a prime and a proper (k, r)-integer, with an error term, $O\left(\frac{n}{\log^H n}\right)$, where $H > 0$.

Let $T(k, r; n)$ denote the number of representations of n as the sum of a prime and a (k, r)-integer. We establish an asymptotic formula for $T(k, r; n)$, which yields (1.1) as a particular case. Further, we improve the O-term in the asymptotic formula for $T(k, r; n)$ from $O\left(\frac{n}{\log^H n}\right)$ to $O(ne^{-B\sqrt{\log n}})$, where B is an absolute positive constant and to $O\left(n^{\frac{9}{10}} \log^{\frac{1}{5}} n\right)$, assuming what we call the page hypothesis (stated below) and the e.R.H., respectively. We obtain the consequent improvements in the O-term of the asymptotic formula for $T^*(k, r; n)$.

Page Hypothesis (cf. [*11*], *p. 117).* The greatest real zero σ possessed by Dirichlet's L-functions with modulus q satisfies the inequality $\sigma < 1 - \frac{A}{\log q}$, where A is an absolute positive constant.

A. Page believes that this hypothesis is very likely to hold (see [11], p. 117) and in fact, he has shown that there is at most one real primitive character which does not satisfy this hypothesis.

2. Prerequisites. The function $\lambda_{k,r}(n)$ introduced by one of the authors and V. C. Harris [14] in some other connection, plays an important role in our present context. $\lambda_{k,r}(n)$ is a multiplicative function defined for powers of an arbitrary prime p as follows:

$$\lambda_{k,r}(p^\alpha) = \begin{cases} 1, & \text{if } \alpha \equiv 0 \pmod{k} \\ -1, & \text{if } \alpha \equiv r \pmod{k} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

For brevity, we here after write $\lambda(n)$ for $\lambda_{k,r}(n)$. It should be noted that $\lambda_{k,r}(n)$ is different from the well-known Liouville's function, which is also generally represented by the same symbol.

It has been shown (cf. [14], Theorem 3) that

$$\sum_{d/n} \lambda(d) = q_{k,r}(n), \tag{2.2}$$

where $q_{k,r}(n) = 1$ or 0 according as $n \in Q_{k,r}$ or $n \notin Q_{k,r}$.

REMARK 2.1. We note for a later use that $\lambda(n) = 0$, unless n is of the form $n = a^k b^r$, in which case $\lambda(n) = \mu(b)$, where μ is the Möbius function.

We now prove the following:

LEMMA 2.1. *For* $x \geq 1$, $n \geq 1$,

$$\sum_{\substack{d \leq x \\ (d,n)=1}} \frac{\lambda(d)}{\varphi(d)} = \prod_{\substack{p \\ p \nmid n}} \left\{1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r}-p^{-k}}{1-p^{-k}}\right\} + O\left(\zeta\left(\frac{k}{r}\right) x^{\frac{1}{r}-\frac{3}{4}}\right), \tag{2.3}$$

where the O-constant is independent of n, k and r, $\zeta(s)$ being the Riemann Zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s > 1$ and $\varphi(n)$ is the Euler totient function.

PROOF. Let $\varepsilon(m) = 1$ or 0 according as $m = 1$ or $m > 1$. Then the series in (2.3) would become $\sum_{d=1}^{\infty} \frac{\lambda(d)\,\varepsilon((d,n))}{\varphi(d)}$.

We have by Remark 2.1 and the fact (cf. [3], theorem 327) that $\Phi(n) > Cn^{\frac{3}{4}}$ for all $n \geq 1$ (C being a suitable positive constant),

$$\sum_{d \leq m} \left| \frac{\lambda(d)\,\varepsilon((d,n))}{\varphi(d)} \right| \leq \sum_{a^k b^r \leq m} \frac{1}{\varphi(a^k b^r)} < \frac{1}{C} \sum_{\substack{a,b \\ a^k b^r \leq m}} (a^k b^r)^{-\frac{3}{4}} <$$

$$< \frac{1}{C} \left(\sum_{a=1}^{\infty} a^{-\frac{3k}{4}} \right) \left(\sum_{b=1}^{\infty} b^{-\frac{3r}{4}} \right) = \frac{1}{C} \zeta\left(\frac{3k}{4}\right) \zeta\left(\frac{3r}{4}\right).$$

Hence the series $\sum_{d=1}^{\infty} \frac{\lambda(d)\,\varepsilon((d,n))}{\varphi(d)}$ is absolutely convergent.

Further, $f(d) = \frac{\lambda(d)\,\varepsilon((d,n))}{\varphi(d)}$ is a multiplicative function of d and $f(1) = 1$.

Hence the series can be expanded into an infinite product of Euler type (cf. [3], Theorem 286), so that by (2.1) we have

$$\sum_{\substack{d=1\\(d,n)=1}} \frac{\lambda(d)}{\varphi(d)} = \prod_{p} \left\{1+\frac{\lambda(p)\varepsilon((p,n))}{\varphi(p)}+\frac{\lambda(p^2)\varepsilon((p^2,n))}{\varphi(p^2)}+\ldots\right\} =$$

$$= \prod_{\substack{p\\p\nmid n}} \left\{1+\frac{1}{\varphi(p^k)}+\frac{1}{\varphi(p^{2k})}+\ldots-\frac{1}{\varphi(p^r)}-\frac{1}{\varphi(p^{r+k})}-\ldots\right\} =$$

$$= \prod_{\substack{p\\p\nmid n}} \left\{1+\frac{1}{p^k(1-p^{-1})}+\frac{1}{p^{2k}(1-p^{-1})}+ \right.$$

$$\left. +\ldots-\left[\frac{1}{p^r(1-p^{-1})}+\frac{1}{p^{r+k}(1-p^{-1})}+\ldots\right]\right\} =$$

$$= \prod_{\substack{p\\p\nmid n}} \left\{1+\frac{1}{1-p^{-1}}\cdot\frac{1}{p^k}\cdot\frac{1}{1-p^{-k}}-\frac{1}{1-p^{-1}}\cdot\frac{1}{p^r}\cdot\frac{1}{1-p^{-k}}\right\} =$$

$$= \prod_{\substack{p\\p\nmid n}} \left\{1-\frac{1}{1-p^{-1}}\cdot\frac{p^{-r}-p^{-k}}{1-p^{-k}}\right\}.$$

Also, by Remark 2.1 and $\varphi(n) > Cn^{\frac{3}{4}}$ for all $n \geq 1$, we have

$$\sum_{\substack{d>x\\(d,n)=1}} \frac{\lambda(d)}{\varphi(d)} \leq \sum_{a^k b^r > x} \frac{1}{\varphi(a^k b^r)} < \frac{1}{C} \sum_{\substack{a,b\\a^k b^r > x}} a^{-\frac{3k}{4}} b^{-\frac{3r}{4}} =$$

$$= \frac{1}{C} \sum_{a=1}^{\infty} a^{-\frac{3k}{4}} \sum_{b > \sqrt[r]{xa^{-k}}} b^{-\frac{3r}{4}} = O\left(\sum_{a=1}^{\infty} a^{-\frac{3k}{4}} (\sqrt[r]{xa^{-k}})^{1-\frac{3r}{4}}\right) =$$

$$= O\left(x^{\frac{1}{r}-\frac{3}{4}} \sum_{a=1}^{\infty} a^{-\frac{k}{r}}\right) = O\left(\zeta\left(\frac{k}{r}\right) x^{\frac{1}{r}-\frac{3}{4}}\right).$$

Since

$$\sum_{\substack{d\leq x\\(d,n)=1}} \frac{\lambda(d)}{\varphi(d)} = \sum_{\substack{d=1\\(d,n)=1}}^{\infty} \frac{\lambda(d)}{\varphi(d)} - \sum_{\substack{d>x\\(d,n)=1}} \frac{\lambda(d)}{\varphi(d)},$$

lemma 2.1 follows.

Let $\pi(x; u, v)$ denote the number of primes $p \leq x$, $p \equiv v \pmod{u}$ and $(u, v) = 1$. The it is well-known that

$$\frac{1}{x}\left\{\pi(x; u, v) - \frac{1}{\varphi(u)} \operatorname{Li} x\right\} \to 0 \text{ as } x \to \infty$$

uniformly with respect to u and v.

Suppose that

$$\pi(x; u, v) = \frac{1}{\varphi(u)} \operatorname{Li} x + O(x\,\omega(x)), \tag{2.4}$$

where $\omega(x)$ *is a monotonic decreasing function tending to zero as* $x \to \infty$, *and the O-constant is independent of u and v.*

There exist explicit expressions for $\omega(x)$ in the literature. As a consequence of SIEGAL–WALFISZ (cf. [17], Hilfssatz 3, p. 598) theorem, we have the following result which is due to VAN DER CORPUT:

LEMMA 2.2. (cf. [16], Footnote 4, pp. 279–280). *Let H be any positive number, then*

$$\pi(x; u, v) = \frac{1}{\varphi(u)} \operatorname{Li} x + O\left(\frac{x}{\log^{5H} x}\right), \tag{2.5}$$

where the O-constant dependes only on H.

LEMMA 2.3. (cf. [11], (33), p. 135). *If the Page hypothesis is true, then*

$$\pi(x; u, v) = \frac{1}{\varphi(u)} \operatorname{Li} x + O\left(x e^{-5B\sqrt{\log x}}\right), \tag{2.6}$$

where B is an absolute positive constant and the O-constant is independent of u and v.

LEMMA 2.4. (cf. [15], Theorem 6, p. 427). *If e.R.H. is true, then for* $u < x$,

$$\pi(x; u, v) = \frac{1}{\varphi(u)} \operatorname{Li} x + O\left(x^{\frac{1}{2}} \log x\right), \tag{2.7}$$

where the O-constant is independent of u and v.

3. Main Results. First we prove the following:

THEOREM 3.1. *If* $T(k, r; n)$ *is the number of representations of n as the sum of a prime and a* (k, r)*-integer, then for* $n \to \infty$,

$$T(k, r; n) = \prod_{\substack{p \\ p \nmid n}} \left\{1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r} - p^{-k}}{1-p^{-k}}\right\} \operatorname{Li} n + O\left(\zeta\left(\frac{k}{r}\right) n [\omega(n)]^{\frac{1}{5}}\right), \tag{3.1}$$

where the O-constant is independent of n, k, r and $\omega(x)$ *is the function which appears in the O-term of* (2.4).

PROOF. Let y denote a certain function of n, to be chosen later, which tends to infinity with n. The O-notation below refers to the passage $n \to \infty$.

We have by (2.2),

$$T(k, r; n) = \sum_{\substack{p \\ p+m=n}} q_{k,r}(m) = \sum_{\substack{p \\ p+m=n}} \sum_{d\delta=m} \lambda(d) = \sum_{\substack{p \\ p+d\delta=n}} \lambda(d) = \tag{3.2}$$

$$= \sum_{\substack{p+d\delta \\ d \leq y}} \lambda(d) + \sum_{\substack{p+d\delta=n \\ d>y}} \lambda(d) = \sum\nolimits_1 + \sum\nolimits_2,$$

say. Now,

(3.3) $$\sum_1 = \sum_{\substack{p+d\delta=n \\ d\le y}} \lambda(d) = \sum_{d\le y} \lambda(d) \sum_{\substack{p<n \\ p\equiv n \pmod d}} 1 =$$

$$= \sum_{\substack{d\le y \\ (d,n)=1}} \lambda(d) \sum_{\substack{p<n \\ p\equiv n \pmod d}} 1 + \sum_{\substack{d\le y \\ (d,n)>1}} \lambda(d) \sum_{\substack{p<n \\ p\equiv n \pmod d}} 1 = \sum_{11} + \sum_{12},$$

say. We now estimate $\sum_{12}$. Since $(d, n) > 1$ and $n = p + d\delta$, we have $(d, n) = p$. It is obvious that the number of d's $\le y$ with $(d, n) = p$ for some prime p, is certainly $\le y$. Hence

(3.4) $$\sum_{12} = \sum_{\substack{d\le y \\ (d,n)>1}} \lambda(d) \sum_{\substack{p<n \\ p\equiv n \pmod d}} 1 = \sum_{\substack{p+d\delta=n \\ d\le y}} \lambda(d) = O(y).$$

Next, by (2.4),

(3.5)
$$\sum_{11} = \sum_{\substack{d\le y \\ (d,n)=1}} \lambda(d) \sum_{\substack{p<n \\ p\equiv n \pmod d}} 1 = \sum_{\substack{d\le y \\ (d,n)=1}} \lambda(d) \left\{ \frac{1}{\varphi(d)} \operatorname{Li} n + O(n\,\omega(n)) \right\} =$$

$$= \operatorname{Li} n \sum_{\substack{d\le y \\ (d,n)=1}} \frac{\lambda(d)}{\varphi(d)} + O(ny\,\omega(n)).$$

Hence, by lemma 2.1, we have

(3.6)
$$\sum_{11} = \prod_{\substack{p \\ p\nmid n}} \left\{ 1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r}-p^{-k}}{1-p^{-k}} \right\} \operatorname{Li} n + O\left(\zeta\left(\frac{k}{r}\right) y^{\frac{1}{r}-\frac{3}{4}} \operatorname{Li} n \right) + O(ny\omega(n)).$$

Also, by Remark 2.1,

(3.7) $$\left|\sum_2\right| = \Big| \sum_{\substack{p+d\delta=n \\ d>y}} \lambda(d) \Big| =$$

$$= \Big| \sum_{\substack{p+a^k b^r \delta=n \\ a^k b^r>y}} \mu(b) \Big| \le \sum_{\substack{p+a^k b^r \delta=n \\ a^k b^r>y}} 1 \le \sum_{\substack{a^k b^r \delta<n \\ a^k b^r>y}} 1 \le \sum_{a^k b^r \delta>y} \frac{n}{a^k b^r} =$$

$$= n \sum_{a=1}^{\infty} a^{-k} \sum_{b>\sqrt[r]{ya^{-k}}} b^{-r} = O\left(n \sum_{a=1}^{\infty} a^{-k} (\sqrt[r]{ya^{-k}})^{1-r} \right) =$$

$$= O\left(ny^{\frac{1}{r}-1} \sum_{a=1}^{\infty} a^{-\frac{k}{r}} \right) = O\left(\zeta\left(\frac{k}{r}\right) ny^{\frac{1}{r}-1} \right).$$

Hence by (3.2), (3.3), (3.4), (3.6) and (3.7), we have

(3.8) $$T(k,r;n) = \prod_{\substack{p\\ p\nmid n}}\left(1-\frac{1}{1-p^{-1}}\cdot\frac{p^{-r}-p^{-k}}{1-p^{-k}}\right)\mathrm{Li}\, n+$$

$$+O\left(\zeta\left(\frac{k}{r}\right)y^{\frac{1}{r}-\frac{3}{4}}\mathrm{Li}\, n\right)+O(ny\,\omega(n))+O(y)+O\left(\zeta\left(\frac{k}{r}\right)y^{\frac{1}{r}-1}\right).$$

Now, choosing $y=[\omega(n)]^{-\frac{4}{5}}$, we see that $y\to\infty$ as $n\to\infty$, since $\omega(n)\to 0$ as $n\to\infty$. The first O-term is (3.8) becomes

$$O\left(\zeta\left(\frac{k}{r}\right)[\omega(n)]^{-\frac{4}{5}\left(\frac{1}{r}-\frac{3}{4}\right)}\mathrm{Li}\, n\right)=O\left(\zeta\left(\frac{k}{r}\right)n[\omega(n)]^{\frac{1}{5}}\right),$$

since $\mathrm{Li}\, n\sim\dfrac{n}{\log n}=O(n)$, $\omega(n)$ is monotonic decreasing and $r\geq 2$.

It is clear that the second and third O-terms in (3.8) are each $O\left(n[\omega(n)]^{\frac{1}{5}}\right)$. The fourth O-term in (3.8) is

$$O\left(\zeta\left(\frac{k}{r}\right)n[\omega(n)]^{-\frac{4}{5}\left(\frac{1}{r}-1\right)}\right)=O\left(\zeta\left(\frac{k}{r}\right)n[\omega(n)]^{\frac{1}{5}}\right),$$

since $\omega(n)$ is monotonic decreasing and $r\geq 2$.

Hence theorem 3.1 follows:

REMARK 3.1 In the evaluation of $\sum_{11}$ above, we argued that

$$\sum_{\substack{d\leq y\\ (d,n)=1}}\lambda(d)\cdot O(n\,\omega(n)) = O(ny\,\omega(n)).$$

This argument holds, provided the O-constant is independent of d. This is indeed true, in virtue of (2.5), (2.6) and (2.7) with

$$\omega(x) = \frac{1}{\log^{5H}x},\ \omega(x) = e^{-5B\sqrt{\log x}}$$

and

$$\omega(x) = x^{-\frac{1}{2}}\log x$$

respectively in (2.4).

In view of the above Remark and $\zeta\left(\frac{k}{r}\right)\leq\zeta\left(\frac{r+1}{r}\right)$, we have the following:

COROLLARY 3.1.

$$(3.9)\qquad T(k, r; n) = \prod_{\substack{p \\ p\nmid n}} \left\{1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r}-p^{-k}}{1-p^{-k}}\right\} \mathrm{Li}\, n + O\left(\frac{n}{\log^H n}\right),$$

where the O-constant depends at most on r and H.

COROLLARY 3.2. *If the Page hypothesis is true, then*

$$(3.10)\qquad T(k, r; n) = \prod_{\substack{p \\ p\nmid n}} \left\{1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r}-p^{-k}}{1-p^{-k}}\right\} \mathrm{Li}\, n + O\left(n e^{-B\sqrt{\log n}}\right),$$

where B is an absolute positive constant and the O-constant depends at most on r.

COROLLARY 3.3. *If the e.R.H. is true, then*

$$(3.11)\qquad T(k, r; n) = \prod_{\substack{p \\ p\nmid n}} \left\{1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r}-p^{-k}}{1-p^{-k}}\right\} \mathrm{Li}\, n + O\left(n^{\frac{9}{10}} \log^{\frac{1}{5}} n\right),$$

where the O-constant depends at most on r.

REMARK 3.2. We note that in the limiting case $k \to \infty$, a (k, r)-integer becomes a r-free integer.

Hence, by taking limits of (3.9), (3.10) and (3.11) as $k \to \infty$, we obtain the following:

$$(3.12)\qquad T(r; n) = \prod_{\substack{p \\ p\nmid n}} \left\{1 - \frac{1}{p^{r-1}(p-1)}\right\} \mathrm{Li}\, n + O\left(\frac{n}{\log^H n}\right),$$

$$(3.13)\qquad T(r; n) = \prod_{\substack{p \\ p\nmid n}} \left\{1 - \frac{1}{p^{r-1}(p-1)}\right\} \mathrm{Li}\, n + O\left(n e^{-B\sqrt{\log n}}\right),$$

$$(3.14)\qquad T(r; n) = \prod_{\substack{p \\ p\nmid n}} \left\{1 - \frac{1}{p^{r-1}(p-1)}\right\} \mathrm{Li}\, n + O\left(n^{\frac{9}{10}} \log^{\frac{1}{5}} n\right).$$

REMARK 3.3. It may be noted that (3.12) is the same as (1.1). However, (3.13) and (3.14) give better O-estimates on the assumption of the Page hypothesis and on the e.R.H. respectively.

COROLLARY 3.4. *If $T^*(k, r; n)$ is the number of representations of n as the sum of a prime and a proper (k, r)-integer, then for $n \to \infty$,*

$$(3.15)\qquad T^*(k, r; n) =$$

$$= \left[\prod_{\substack{p \\ p\nmid n}} \left\{1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r}-p^{-k}}{1-p^{-k}}\right\} - \prod_{\substack{p \\ p\nmid n}} \left\{1 - \frac{1}{p^{r-1}(p-1)}\right\}\right] \mathrm{Li}\, n + O\left(\frac{n}{\log^H n}\right),$$

where the O-constant depends at most on r and H.

PROOF. This follows by (3.9) and (3.12), since

$$T^*(k, r; n) = T(k, r; n) - T(r; n).$$

COROLLARY 3.5. *The O-term in* (3.15) *can be replaced by* $O(ne^{-B\sqrt{\log n}})$ *and by* $O\left(n^{\frac{9}{10}} \log^{\frac{1}{5}} n\right)$, *on the assumption of the page hypothesis and on the e.R.H. respectively.*

THEOREM 3.2. *Every sufficiently large integer can be expressed as the sum of a prime and a proper* (k, r)*-integer.*

PROOF. Let

$$\alpha(k, r; p) = 1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r}-p^{-k}}{1-p^{-k}} \quad \text{and} \quad \beta(r; p) = 1 - \frac{1}{p^{r-1}(p-1)}.$$

It is clear that $\alpha(k, r; p) > \beta(r; p)$. By (3.15), we have

$$T^*(k, r; n) = \left[\prod_{\substack{p \\ p\nmid n}} \alpha(k, r; p) - \prod_{\substack{p \\ p\nmid n}} \beta(r; p)\right] \mathrm{Li}\, n + O\left(\frac{n}{\log^H n}\right), \tag{3.16}$$

where the O-constant depends at most on r and H.

Since the main term in (3.16) is positive for every n, it is sufficient, if we prove the following: As $n \to \infty$,

$$T^*(k, r; n) \sim \left[\prod_{\substack{p \\ p\nmid n}} \alpha(k, r; p) - \prod_{\substack{p \\ p\nmid n}} \beta(r; p)\right] \mathrm{Li}\, n. \tag{3.17}$$

Let

$$\gamma(k, r; p) = \frac{\alpha(k, r; p)}{\beta(r; p)} - 1,$$

we have

$$\beta(r, p;) \cdot \gamma(k, r; p) = \alpha(k, r; p) - \beta(r; p) =$$

$$= \frac{1}{p^r(1-p^{-1})} - \frac{p^{-r}-p^{-k}}{(1-p^{-1})\cdot(1-p^{-k})} = \frac{1}{p^r(1-p^{-1})} \cdot \frac{p^r-1}{p^k-1} > \frac{1}{2p^k},$$

since $p^r - 1 > \frac{p^r}{2}$, $\frac{1}{1-p^{-1}} > 1$ and $\frac{1}{p^k-1} > \frac{1}{p^k}$.

Hence $\gamma(k, r; p) > \frac{1}{\beta(r; p)} \cdot \frac{1}{2p^k}$, so that

$$\frac{\alpha(k, r; p)}{\beta(r; p)} = 1 + \gamma(k, r; p) > 1 + \frac{1}{2p^k}.$$

We note that $\prod_p \beta(r; p)$, the product being extended over all primes p, is a convergent infinite product, and we denote it by β_r.

We have

$$\prod_{\substack{p \\ p\nmid n}} \alpha(k, r; p) - \prod_{\substack{p \\ p\nmid n}} \beta(r; p) =$$

$$= \prod_{\substack{p \\ p\nmid n}} \beta(r; p)\left[\prod_{\substack{p \\ p\nmid n}} \frac{\alpha(k, r; p)}{\beta(r; p)} - 1\right] > \prod_{\substack{p \\ p\nmid n}} \beta(r; p)\left[\prod_{\substack{p \\ p\nmid n}} \left(1 + \frac{1}{2p^k}\right) - 1\right] =$$

$$= \frac{\prod_{p} \beta(r; p)}{\prod_{p\mid n} \beta(r; p)} \left[\prod_{\substack{p \\ p\nmid n}} \left(1 + \frac{1}{2p^k}\right) - 1\right] > \beta_r \left[\prod_{\substack{p \\ p\nmid n}} \left(1 + \frac{1}{2p^k}\right) - 1\right],$$

since $\prod_{p\mid n} \beta(r; p) < 1$.

Hence

$$\prod_{\substack{p \\ p\nmid n}} \alpha(k, r; p) - \prod_{\substack{p \\ p\nmid n}} \beta(r; p) > \beta_r \cdot \frac{1}{2p^k},$$

where the p on the right is any prime not dividing n.

As a simple consequence of the prime number theorem in the form $\sum_{p\le x} \log p \sim x$, we see that for every sufficiently large n, there is a prime p such that $p \le 2 \log n$ and $p\nmid n$. Hence for all sufficiently large n, we have

$$\prod_{\substack{p \\ p\nmid n}} \alpha(k, r; p) - \prod_{\substack{p \\ p\nmid n}} \beta(r; p) > \frac{\beta_r}{2} \cdot \frac{1}{2^k \log^k n} = \frac{\beta_r}{2^{k+1} \log^k n}.$$

This together with (3.16) gives

$$\frac{T^*(k, r; n)}{\left[\prod_{\substack{p \\ p\nmid n}} \alpha(k, r; p) - \prod_{\substack{p \\ p\nmid n}} \beta(r; p) \operatorname{Li} n\right]} = 1 + O\left(\frac{n}{\log^H n} \cdot \frac{2^{k+1} \log^k n}{\beta_r} \cdot \frac{\log n}{n}\right) =$$

$$= 1 + O\left[\frac{1}{(\log n)^{H-k-1}}\right],$$

where the O-constant is independent of n.

Hence, if $H > k+1$, then (3.17) follows. Since the number H in lemma 2.2 is an arbitrary positive number, we can take H such that $H > k+1$ for any fixed k.

Thus theorem 3.2 follows.

REMARK 3.4. On the assumption of the Page hypothesis or on the e.R.H. also, the conclusion of theorem 3.2 follows. This is rather more immediate in view of Corollary 3.5. However, we have the improved O-estimates in the asymptotic formula for $T^*(k, r; n)$ on the assumption of these hypotheses.

We have

$$\prod_{\substack{p \\ p\nmid n}} \alpha(k, r; p) - \prod_{\substack{p \\ p\nmid n}} \beta(r; p) =$$

$$= \prod_{\substack{p \\ p\nmid n}} \beta(r; p)\left[\prod_{\substack{p \\ p\nmid n}} \frac{\alpha(k, r; p)}{\beta(r; p)} - 1\right] > \prod_{\substack{p \\ p\nmid n}} \beta(r; p)\left[\prod_{\substack{p \\ p\nmid n}} \left(1 + \frac{1}{2p^k}\right) - 1\right] =$$

$$= \frac{\prod_{p} \beta(r; p)}{\prod_{p\mid n} \beta(r; p)}\left[\prod_{\substack{p \\ p\nmid n}} \left(1 + \frac{1}{2p^k}\right) - 1\right] > \beta_r\left[\prod_{\substack{p \\ p\nmid n}} \left(1 + \frac{1}{2p^k}\right) - 1\right],$$

since $\prod_{p\mid n} \beta(r; p) < 1$.

Hence

$$\prod_{\substack{p \\ p\nmid n}} \alpha(k, r; p) - \prod_{\substack{p \\ p\nmid n}} \beta(r; p) > \beta_r \cdot \frac{1}{2p^k},$$

where the p on the right is any prime not dividing n.

As a simple consequence of the prime number theorem in the form $\sum_{p \le x} \log p \sim x$, we see that for every sufficiently large n, there is a prime p such that $p \le 2 \log n$ and $p \nmid n$. Hence for all sufficiently large n, we have

$$\prod_{\substack{p \\ p\nmid n}} \alpha(k, r; p) - \prod_{\substack{p \\ p\nmid n}} \beta(r; p) > \frac{\beta_r}{2} \cdot \frac{1}{2^k \log^k n} = \frac{\beta_r}{2^{k+1} \log^k n}.$$

This together with (3.16) gives

$$\frac{T^*(k, r; n)}{\left[\prod_{\substack{p \\ p\nmid n}} \alpha(k, r; p) - \prod_{\substack{p \\ p\nmid n}} \beta(r; p) \operatorname{Li} n\right]} = 1 + O\left(\frac{n}{\log^H n} \cdot \frac{2^{k+1} \log^k n}{\beta_r} \cdot \frac{\log n}{n}\right) =$$

$$= 1 + O\left[\frac{1}{(\log n)^{H-k-1}}\right],$$

where the O-constant is independent of n.

Hence, if $H > k+1$, then (3.17) follows. Since the number H in lemma 2.2 is an arbitrary positive number, we can take H such that $H > k+1$ for any fixed k.

Thus theorem 3.2 follows.

REMARK 3.4. On the assumption of the Page hypothesis or on the e.R.H. also, the conclusion of theorem 3.2 follows. This is rather more immediate in view of Corollary 3.5. However, we have the improved O-estimates in the asymptotic formula for $T^*(k, r; n)$ on the assumption of these hypotheses.

As particular cases of theorem 3.2, we have the following:

COROLLARY 3.6. ($r = 2$). *Every sufficiently large integer can be expressed as the sum of a prime and an integer of the form* $a^k b$, *where* $a > 1$ *and* b *is a square-free integer;* k *being any given integer* ≥ 3.

COROLLARY 3.7. ($k = 3$, $r = 2$). *Every sufficiently large integer can be expressed as the sum of a prime and an integer of the form* $a^3 b$, *where* $a > 1$ *and* b *is square-free.*

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