ON AN EXTENSION OF NAGELL'S TOTIENT FUNCTION AND SOME APPLICATIONS

M.V. Subbarao
University of Alberta
Edmonton, Alberta T6G 2G1
Canada

V.V. Subrahmanya Sastri
SSS Institute of Higher Learning
Anantapur, A.P. 515001
India

e-mail: m.v.subbarao@ualberta.ca

1. Introduction

In 1923 Nagell [4] studied the totient function $\theta(m,n)$ for positive integers m,n defined as the number of positive integers $x \leq n$ for which $(x,n) = 1 = (m-x,n)$. Several interesting properties and applications of the same were studied.

We shall denote by $(\text{mod}^* n; m)$, the set of all positive integers $x \leq n$ for which $(x,n) = 1 = (m-x,n)$ and call it an RRS (Reduced Residue System) $(\text{mod}\, n; m)$. Note that $(\text{mod}^*\, n; n)$ is simply a reduced residue system $\text{mod}\, n$, and is denoted by $(\text{mod}^*\, n)$. As usual we shall denote a Complete Reduced System (CRS) $\text{mod}\, n$ by simply writing $(\text{mod}\, n)$. In general, we define for $r \geq 2$, $\theta(m_1, m_2, \ldots, m_r; n)$ as the number of positive integers $x \leq n$ for which $(x,n) = 1 = (m_i - x, n)$, $i = 1, 2, \ldots, r$, where m_i, $i = 1, 2, \ldots, r$ are positive integers, and denote analogously this RRS $(\text{mod}^* n : m_1, m_2, \ldots, m_r)$. As the referee suggested, in order not to confuse this with a congruence relation, for example, say $x = n(\text{mod}\, m)$, we simply write $r = n \bmod m$, omitting parentheses. In this paper, we study the function $\theta(m_1, m_2 \ldots, m_r; n)$ and obtain some of its arithmetical properties and identities that appear to be new. For the sake of simplicity we restrict ourselves mostly to the case $r = 2$. As an application of this function, we construct the Ramanujan Sum analogues associated with this function and study

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their properties. We obtain also some results involving an associated zeta function analogue, and obtain applications to certain restricted relative partitions mod N.

2. Preliminaries

Let $\mu(n)$ denote the well known Mõbius function given by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes } (k \geq 0) \\ 0 & \text{otherwise.} \end{cases}$$

We recall that an arithmetic function f is said to be multiplicative if

$$f(mn) = f(m)f(n) \quad \text{whenever} \quad (m,n) = 1.$$

We say that f is completely multiplicative if this multiplicative property holds for all m, n.

Further, if

$$e_n(x) \stackrel{\text{def}}{\equiv} \exp\{2\pi i x/n\}$$

then

$$\sum_{x \pmod n} e_n(xd) = \begin{cases} n & \text{if } n|d; \\ 0 & \text{otherwise.} \end{cases}$$

The Ramanujan sum $C(\ell, n)$ (see [3]) and some other arithmetic functions that are needed in this paper are defined below.

$$C(\ell, n) \stackrel{\text{def}}{\equiv} \sum_{x \pmod{^* n}} e_n(x\ell)$$

$$I_k(n) \stackrel{\text{def}}{\equiv} n^k, \qquad \forall\, n$$

$$E(n) = I_0(n) = 1, \qquad \forall\, n$$

$$E_0(n) = [1/n] = \begin{cases} 1, & n = 1 \\ 0, & n > 1. \end{cases}$$

$$\mu(m_1, m_2; n) \stackrel{\text{def}}{\equiv} \begin{cases} (-1)^{w_1(n)}(-2)^{w_2(n)}(-3)^{w_3(n)}, & \\ \quad \text{when } n \text{ is a product} & \\ \quad \text{of distinct primes}, & \\ 0, \quad \text{otherwise}. & \end{cases}$$

$$\lambda(m_1, m_2; n) \stackrel{\text{def}}{\equiv} 2^{\Omega_2(n)} 3^{\Omega_3(n)},$$

where

$$w_1(n) = \# \quad \{\text{distinct primes} \quad p : p \mid (n, m_1, m_2)\}$$

$$w_2(n) = \# \quad \{\text{distinct primes} \quad p : p \mid n \quad \text{and} \quad p \mid \quad \text{just one of} \quad m_1, m_2, m_1 - m_2\}$$

$$w_3(n) = \# \quad \{\text{distinct primes} \quad p : p \mid n \quad \text{and} \quad p \nmid \quad \text{any one of} \quad m_1, m_2, m_1 - m_2\}$$

and

$\Omega_i(n),\ i = 1, 2, 3$ denote the total number of prime factors of n corresponding to $w_i(n),\ i = 1, 2, 3$ respectively.

Let us call the primes of the above three types as primes of type 1, type 2 and type 3 respectively with respect to the pair of integers m_1, m_2.

If $f(n)$, $g(n)$ are any two arithmetic functions of n (where f or g or both may also be functions of some more parameters), we denote by $\circ$ the Dirichlet Convolution of f and g with respect to n, for e.g. $I(n) \circ \mu(m_1, m_2; n) =$ Dirichlet product of $I(n)$ and $\mu(m_1, m_2; n)$ with respect to the argument n.

3. Formulae for $\theta(m_1, m_2; n)$

We first obtain a 'Möbius type' inversion formula given by

3.1. Theorem. *Let $f(x)$ be a periodic function of a real variable x with period 1 and*

$$F(m_1, m_2; n) = \sum_{r(\text{mod}^* n;\, m_1, m_2)} f(r/n),$$

then

$$(3.2) \qquad F(m_1, m_2; n) = \sum \mu(d_1)\mu(d_2) G(d_1, d_2; m_1, m_2; n),$$

where the summation is over those divisor pairs d_1, d_2 of n for which $(d_i, m_i) = 1$, $i = 1, 2$ and $(d_1, d_2) | (m_1 - m_2)$ and

$$G(d_1, d_2; m_1, m_2; n) = \sum_{r(\text{mod}^* n),\, r \equiv m_i (\text{mod}\, d_i),\, i=1,2} f(r/n)$$

This is easily proved by the inclusion-exclusion combinatorial principle.

In particular, choosing $f(x) = 1$, we have

3.3. Theorem. $\theta(m_1, m_2; n) = \sum \mu(d_1)\mu(d_2)\phi(n)/\phi(\delta)$ *where $\delta =$ ℓ.c.m. $\{d_1, d_2\}$ and the summation is as in (3.2).*

The proof follows easily by making use of the following well known result of
Vaidyanathaswamy [7].

3.4. If $d | N$ and $(t, d) = 1$ then in any RRS $(\text{mod}\, N)$ there are $\phi(N)/\phi(d)$ integers congruent to $t (\text{mod}\, d)$.

When $f(x) = 1$, Theorem 3.1 gives

$$G(d_1, d_2; m_1, m_2; n) = \begin{cases} \phi(n)/\phi(\delta) & \text{if} \quad (d_1, d_2) \,|\, (m_1 - m_2) \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta = \ell.\text{c.m.}\ \{d_1, d_2\}$ by considering the simultaneous solutions of $r \equiv m_i \pmod{d_i}$, $i = 1, 2$.

Next we note

3.5. Lemma. *Let* $f \neq 0$, g, h, s *be multiplicative arithmetic functions and for any three positive integers* n, m_1, m_2, *let*

$$A = \{\textit{ordered pairs of positive integers } \langle d_1, d_2 \rangle \textit{ such that } d_1, d_2 \mid n, \\ (d_i, m_i) = 1, \quad i = 1, 2 \text{ and } d = (d_1, d_2) \mid (m_1 - m_2)\}.$$

Then

$$H(m_1, m_2; n) \stackrel{\text{def}}{\equiv} s(n) \sum_A g(d_1)\, h(d_2) / f(d_1 d_2 / d)$$

is multiplicative in n.

In particular, for $g = h = \mu$, $f = \phi$, $s = E$, *we obtain that* $\theta(m_1, m_2; n)$ *is multiplicative in* n.

The result follows easily following the usual procedure of splitting each of the two divisors d_j $(j = 1, 2)$ of $n = n_1 n_2$ with $(n_1, n_2) = 1$ and satisfying conditions in A into product of two relatively prime numbers d_{j1}, d_{j2} dividing n_1 and n_2 respectively and using the multiplicativity of f, g, h and s.

In the particular case mentioned in 3.5, we obtain through evaluation of θ at prime powers that

$$\begin{aligned} \theta(m_1, m_2; n) &= \phi(n) \prod_p \left(1 - \frac{1}{\phi(p)}\right) \prod_q \left(1 - \frac{2}{\phi(q)}\right) \\ &= n \prod_u \left(1 - \frac{1}{u}\right) \prod_p \left(1 - \frac{2}{p}\right) \prod_q \left(1 - \frac{3}{q}\right), \end{aligned}$$

where u, p and q run through the prime divisors of n, of types 1, 2 and 3 respectively.

Note that we have

$$\theta(m_1, m_2; n) = I(n) \circ \mu(m_1, m_2; n). \tag{3.6}$$

Remark. More generally, it can be proved by using a similar argument that

$$\theta(m_1, m_2, \ldots, m_r; n) = \sum \mu(d_1)\mu(d_2)\cdots\mu(d_r)\phi(n)/\phi(\delta),$$

where the summation is over those r-tuples $\langle d_1, d_2, \ldots, d_r\rangle$ for which

(i) $d_j \mid n$ and $(d_j, m_j) = 1, \quad j = 1, 2, \ldots, r$

(ii) $(d_i, d_j) \mid (m_i - m_j), \quad 1 \le i < j \le r, \quad j = 1, 2, \ldots, r$ and

(iii) $\delta = \{d_1, d_2, \ldots, d_r\}$, the ℓ.c.m. of $d_1, d_2, \ldots, d_r$.

The function θ is multiplicative in n and so is given by

$$\theta(m_1, m_2, \ldots, m_r; n) = n\prod_{p_1}\left(1 - \frac{1}{p_1}\right)\prod_{p_2}\left(1 - \frac{2}{p_2}\right)\cdots\prod_{p_{r+1}}\left(1 - \frac{r+1}{p_{r+1}}\right),$$

wherein for $t = 1, 2, \ldots, (r+1)$, p_t runs over those prime divisors of n which are relatively prime to just $r + \binom{r}{2} - \binom{r+2-t}{2}$ of the members of the set $M = \{m_j, (m_i - m_j) : 1 \le i < j \le r, \; j = 1, 2, \ldots, r\}$, that is, those which just divide $\binom{r+2-t}{2}$ members of the set M.

4. Some Identities Involving $\theta(m_1, m_2; n)$

We define a zeta function analogue given by

$$\begin{aligned}\zeta(m_1, m_2; s) &\overset{\text{def}}{\equiv} \sum_{n=1}^{\infty} \lambda(m_1, m_2; n)/n^s \\ &= \prod_u \left(1 + u^{-s} + u^{-2s} + \ldots\right)\prod_p \left(1 + 2p^{-s} + 2^2p^{-2s} + \ldots\right) \\ &\quad \times \prod_q \left(1 + 3q^{-s} + 3^2q^{-2s} + \ldots\right),\end{aligned}$$

which is convergent for $\sigma = Rl\, s > 1$, since

$$\Big|\sum_u \sum_j u^{-js} + \sum_p \sum_j 2^j p^{-js} + \sum_q \sum_j 3^j q^{-js}\Big| \le \sum_n 3^{w(n)} n^{-\sigma},$$

where $w(n) = w_1(n) + w_2(n) + w_3(n)$.

Then we have from (3.6) that

$$(4.1) \qquad \sum_{n=1}^{\infty} \theta(m_1, m_2; n)/n^s = \zeta(s-1)/\zeta(m_1, m_2; s), \quad Rl\, s > 2.$$

We further note that $\lambda(m_1, m_2; n) \circ \mu(m_1, m_2; n) = E_o(n)$ and so

$$\sum_{n=1}^{\infty} \mu(m_1, m_2, n)/n^s = \prod_u (1 - u^{-s}) \prod_p (1 - 2p^{-s}) \prod_q (1 - 3q^{-s})$$
$$= M(m_1, m_s; s), \quad \text{say,}$$

which is convergent for $\sigma = Rl\, s > 1$ since

$$\Big|\sum u^{-s} + \sum 2p^{-s} + \sum 3q^{-s}\Big| \le 3 \sum n^{-\sigma}.$$

Hence we also have for $Rl\, s > 1$

$$(4.2) \qquad \sum_{n=1}^{\infty} \lambda(m_1, m_2; n)/n^s = 1/M(m_1, m_2; s) = \zeta(m_1, m_2; s).$$

Let

$$(4.3) \qquad \tau(m_1, m_2; n) \stackrel{\text{def}}{\equiv} E(n) \circ \lambda(m_1, m_2; n)$$

and

(4.4)

$$\sigma^{(k)}(m_1, m_2; n) \stackrel{\text{def}}{\equiv} \sum_{d|n} 2^{\Omega_2(d)} 3^{\Omega_3(d)} (n/d)^k = I_k(n) \circ \lambda(m_1, m_2; n).$$

Then τ is a weighted divisor function of n, with a weight $2^{\Omega_2(n/d)}3^{\Omega_3(n/d)}$ attached to each divisor d of n. $\sigma^{(k)}(m_1, m_2; n)$ is the corresponding weighted sum of the k-th powers of divisors of n. We then have

$$\sum_1^\infty \tau(m_1, m_2; n)/n^s = \zeta(s)\zeta(m_1, m_2, s), \qquad Rl\, s > 1 \tag{4.5}$$

$$\sum_1^\infty \sigma^{(k)}(m_1, m_2; n)/n^s = \zeta(s-k)\zeta(m_1, m_2; s), \qquad Rl\, s \geq 1, \quad k \geq 1. \tag{4.6}$$

These identities can be easily verified. We further have the following multiplicative and summatory results.

$$\theta(m_1, m_2; n_1 n_2)\theta(m_1, m_2, (n_1, n_2)) = (n_1, n_2)\theta(m_1, m_2; n_1)\theta(m_1, m_2; n_2) \tag{4.7}$$

(4.8)
$$\theta(m_1, m_2; \{n_1, n_2\})\theta(m_1, m_2; (n_1, n_2)) = \theta(m_1, m_2; n_1)\theta(m_1, m_2; n_2)$$

$$\sum_{d|n} \theta(m_1, m_2; d)\mu(n/d) = \theta(m_1, m_2; n)\phi(n)/n \tag{4.9}$$

(4.10)
$$\sum_{d|n} \theta(m_1, m_2; d)\mu(m_1, m_2; n/d) = \{\theta(m_1, m_2; n)\}^2/n.$$

We shall indicate the proofs of (4.7) and (4.9).

Proof of (4.7). Since the functions are multiplicative, it is enough to prove the result when $n_1 = p^{\alpha_1}$, $n_2 = p^{\alpha_2}$ (prime powers).

If $p_1 \neq p_2$, using the multiplicativity of θ, we have

$$\begin{aligned}&\theta(m_1, m_2; p_1^{\alpha_1} p_2^{\alpha_2})\theta(m_1, m_2; (p_1^{\alpha_1}, p_2^{\alpha_2}))\\&\quad = \theta(m_1, m_2; p_1^{\alpha_1})\theta(m_1, m_2; p_2^{\alpha_2})\theta(m_1, m_2, 1)\\&\quad = \theta(m_1, m_2; p_1^{\alpha_1})\theta(m_1, m_2; p_2^{\alpha_2})(p_1^{\alpha_1}, p_2^{\alpha_2}).\end{aligned}$$

If $p_1 = p_2$ and p_1 divides both m_1 and m_2 we have $(p_1^{\alpha_1}, p_2^{\alpha_2}) = p_1^{\min(\alpha_1, \alpha_2)}$ and so

$$\begin{aligned}&\theta(m_1, m_2; p_1^{\alpha_1} p_1^{\alpha_2})\theta(m_1, m_2; (p_1^{\alpha_1}, p_1^{\alpha_2}))\\&\quad = \theta(m_1, m_2; p_1^{\alpha_1+\alpha_2})\theta(m_1, m_2; p_1^{\min(\alpha_1, \alpha_2)})\\&\quad = p_1^{\alpha_1+\alpha_2}(1 - 1/p_1)p_1^{\min(\alpha_1, \alpha_2)}(1 - 1/p_1)\\&\quad = \theta(m_1, m_2; p_1^{\alpha_1})\theta(m_1, m_2; p_2^{\alpha_2})(p_1^{\alpha_1}, p_2^{\alpha_2})\\&\qquad \text{since} \quad p_1 = p_2.\end{aligned}$$

The results in the other cases follow similarly.

Proof of (4.9). For $n = p^\alpha$, we have

$$\begin{aligned}\sum_{\beta=0}^{\alpha} \theta(m_1, m_2; p^\beta)\mu(p^{\alpha-\beta}) &= \theta(m_1, m_2; p^\alpha) - \theta(m_1, m_2; p^{\alpha-1})\\&= \theta(m_1, m_2; p^\alpha)(1 - 1/p)\\&= \theta(m_1, m_2; p^\alpha)\phi(p^\alpha)/p^\alpha.\end{aligned}$$

5. Allied Ramanujan Sum Analogues

We define two Ramanujan sum analogues:

$$C(m_1, m_2, \ell, n) \stackrel{\text{def}}{\equiv} \sum_{r(\text{mod}^* \, n; m_1, m_2)} e_n(\ell r) \tag{5.1}$$

and

$$(5.2)\qquad \widetilde{C}(m_1,m_2;\ell,n) \stackrel{\text{def}}{\equiv} \sum_{d|(\ell,n)} d\mu(m_1,m_2;n/d).$$

It is easy to see that C is modular or periodic in each of m_1, m_2 and $\ell(\bmod n)$ since $(n, m_i + kn - r) = (n, m_i - r)$ and $e((\ell+kn)r) = e_n(\ell r)$. Further, since $((\ell,n),n) = (\ell,n)$, $\widetilde{C}$ is an even function of $\ell(\bmod n)$. We refer to E. Cohen [2] for the definition and properties of even functions. Also whenever $(n_1,n_2)=1$, we have $(\ell, n_1 n_2) = (\ell,n_1)(\ell,n_2)$, where $((\ell,n_1),(\ell,n_2)) = 1$ and so $\widetilde{C}$ is multiplicative in n.

We also have a translation property of C given by

5.3. Theorem. *If* $(m,n)=1$, *then*

$$C(m_1,m_2;\ell m,n) = C(m_1 m, m_2 m;\ell,n).$$

This follows on noting that when $(m,n)=1$, r runs $(\bmod^* n; m_1,m_2)$ if and only if mr runs $(\bmod^* n; mm_1, mm_2)$.

In the particular case when $n\,|\,\ell$, we have

5.3.1. Corollary. *If* $(m,n)=1$, *then*

$$\theta(m_1,m_2;n) = \theta(m_1 m, m_2 m; n).$$

When $n\,|\,m_1$ and m_2 in (5.3) we have

5.3.2. Corollary. *When* $(m,n)=1$ *the Ramanujan sum* $C(\ell,n)$ *satisfies* $C(\ell m,n) = C(\ell,n)$.

5.4. Theorem. *Whenever* $(n_1,n_2)=1$, *we have*

$$C(m_{11},m_{12};\ell_1,n_1)C(m_{21},m_{22};\ell_2,n_2) = C(M_1,M_2;\ell,n_1 n_2),$$

where $\ell = \ell_1 n_2 + \ell_2 n_1$, $M_1 = m_{11}n_2 + m_{21}n_1$ and $M_2 = m_{12}n_2 + m_{22}n_1$.

Proof. Let $r_1(\text{mod}^* n_1; m_{11}, m_{12})$ and $r_2(\text{mod}^* n; m_{21}, m_{22})$ and $s = n_1 r_2 + n_2 r_1$. Since $(n_1, n_2) = 1$ we have $(r_1, n_1) = 1$ and $(r_2, n_2) = 1 \Longrightarrow (s, n_1 n_2) = 1$. Again because of the same reason

$$(m_{1i} - r_1, n_1) = 1,\ (m_{2i} - r_2, n_2) = 1 \Longrightarrow (M_i - s, n_1 n_2) = 1,\ i = 1, 2.$$

So, also for $s(\text{mod}^* n_1 n_2; M_1, M_2)$ and this proves the theorem since

$$\begin{aligned}
&C(m_{11}, m_{12}; \ell_1, n_1) C(m_{21}, m_{22}; \ell_2, n_2) \\
&= \sum_{r_i(\text{mod}^* n_i; m_{i1}, m_{i2}), i=1,2} \exp\{2\pi i(\ell_1 r_1 n_2 + \ell_2 r_2 n_1)/n_1 n_2\} \\
&= \sum_{s(\text{mod}\, n_1 n_2; M_1, M_2)} \exp(2\pi \ell s/n_1 n_2) \\
&= C(M_1, M_2; \ell, n_1 n_2).
\end{aligned}$$

We also have an analogue of a result of Ramanujan.

5.5. Theorem.

$$\sigma^{(s)}(\ell) = \ell^s \zeta(m_1, m_2; s+1) \sum_{n=1}^{\infty} \widetilde{C}(m_1, m_2; n)/n^{s+1}, \quad R\ell\ s > 0,$$

where $\sigma^{(s)}(\ell) =$ *sum of the* s^{th} *powers of the positive divisors of* ℓ.

This follows on noting that we can write (5.2) as $\widetilde{C}(m_1, m_2; \ell, n) = I(\ell, n) \circ \mu(m_1, m_2; n)$ where

$$I(\ell, n) = \begin{cases} n & \text{if } n \mid \ell \\ 0 & \text{otherwise,} \end{cases}$$

on realizing that $\sum_{n=1}^{\infty} I(\ell, n)/n^s = \sigma^{(1-s)}(\ell) = \sigma^{(s-1)}(\ell)/\ell^{s-1}$.

5.6. Theorem. *Hölder type identity for* $C(m_1, m_2; \ell, n)$. *Whenever* $d \mid (\ell, n)$, *we have*

$$C(m_1, m_2; \ell, n) = \theta(m_1, m_2; n) C(m_1, m_2; \ell/d, n/d)/\theta(m_1, m_2; n/d).$$

This follows from the definition of C, using the

5.7. Lemma. *For any given divisor* d *of* n, *and any given* j *belonging to the residue system* $(\text{mod}^* d; m_1, m_2)$ *there are* $\theta(m_1, m_2; n)/\theta(m_1, m_2; d)$ *numbers congruent to* $j \pmod d$ *in the residue system* $(\text{mod}^* n; m_1, m_2)$.

This lemma is easily proved on the same lines as result (3.6) above of
Vaidyanathaswamy [7].

5.8. Corollary. *When* $g = (\ell, n)$,

$$C(m_1, m_2; \ell, n)$$
$$= \theta(m_1, m_2; n) C(m_1 \ell/g, m_2 \ell/g; 1, n/g)/\theta(m_1, m_2; n/g).$$

This follows from Lemma (5.7) and Theorem (5.3).
Next we shall obtain some identities for $\widetilde{C}(m_1, m_2; \ell, n)$.

5.9. Theorem. *If* $g = (\ell, n)$, *then the identity*

$$\widetilde{C}(m_1, m_2; \ell, n) = \theta(m_1, m_2; n) \mu(m_1, m_2; n/g)/\theta(m_1, m_2; n/g)$$

holds under the following conditions.

(i) *For all* m_1, m_2 *when* $(n, 6) = 1$
(ii) *For those* m_1, m_2 *with respect to which* 2 *is not of type* 2, *whenever* $2|n$ *and*
(iii) *For those* m_1, m_2 *with respect to which* 3 *is not of type* 3, *whenever* $3|n$.

This is a particular case of the following

5.10. Theorem. *Let f be a completely multiplicative function and let $A(n) = \mu(m_1, m_2; n)h(n)$, where $h(n)$ is a multiplicative function. Then the sum*

$$\widetilde{s}(m_1, m_2; \ell, n) \overset{\text{def}}{=} \sum_{d|g} f(d)A(n/d), \quad g = (\ell, n)$$

satisfies the identity

$$\widetilde{s}(m_1, m_2; \ell, n) = F(m_1, m_2; n)A(n/g)/F(m_1, m_2; n/g)$$

where

$$\begin{aligned} F(m_1, m_2; n) &= (f \circ A)(n) \\ &= f(n)\prod_{p|n}\left(1 + \frac{\mu(m_1, m_2; p)h(p)}{f(p)}\right) \end{aligned}$$

provided that

(i) $f(p) \neq 0$, *for all* $p|n$
(ii) $f(p) \neq h(p)$ *for* $p|n$ *of type 1*
(iii) $f(p) \neq 2h(p)$ *for* $p|n$ *of type 2 and*
(iv) $f(p) \neq 3h(p)$ *for* $p|n$ *of type 3.*

(Note that $A(n)$ is actually a function of m_1, m_2 and n) This theorem is a generalization of Theorem 8.8, pp. 163-164 of Apostol [1].

Proof. We first note that

$$\widetilde{s}(m_1, m_2; \ell, n) = \sum_{d|g} f(d)\mu(m_1, m_2; n/d)h(n/d)$$

(noting that $n/d = (n/g)(g/d)$ has a square factor whenever

$(n/g, g/d) \neq 1$ and using the definition of μ in the preceding step), we have with $g/d = \delta$ that

$$\widetilde{s}(m_1, m_2; \ell, n) = \sum_{\delta | g, (\delta, n/g)=1} f(g/\delta)\mu(m_1, m_2; \delta n/g)h(\delta n/g)$$

$$= f(g)\mu(m_1, m_2; n/g)h(n/g) \sum_{\delta | g, (\delta, n/g)=1} \mu(m_1, m_2; \delta)h(\delta)/f(\delta)$$

(using the complete multiplicativity of f, satisfying (i) to (iv) and definition of μ)

$$= f(g)A(n/g) \prod_{p|g, p\nmid n/g} \left(1 + \frac{\mu(m_1, m_2; p)h(p)}{f(p)}\right).$$

But

$$F(m_1, m_2; n) = \sum_{d|n} f(d)\mu(m_1, m_2; n/d)h(n/d)$$

$$= f(n)\sum_{e|n} \mu(m_1, m_2; e)h(e)/f(e), \quad (e = n/d)$$

$$= f(n)\prod_{p|n}(1 + \mu(m_1, m_2; p)h(p)/f(p)),$$

in view of multiplicativity of μ and h.

Hence we obtain the theorem using the complete multiplicativity of f.

When we choose $f(n) = n$ and $h(n) = 1$ for all n, in the above Theorem 5.9 follows.

5.11. Theorem. *A Brauer-Rademacher type identity holds for $\widetilde{C}(m_1, m_2; \ell, n)$ under the conditions of Theorem 5.9. It is given* *by*

$$\theta(m_1, m_2; n) \sum_{d|n, (\ell, d)=1} d\mu(n/d)/\theta(m_1, m_2; d)$$

$$= \widetilde{C}(m_1, m_2; \ell, n)\mu(n)\mu(n/g)\lambda(m_1, m_2; n/g)/\mu(m_1, m_2; n/g)$$

where $g = (\ell, n)$.

Proof. Defining $f(n) = n/\theta(m_1, m_2; n)$,
$h(n) = \lambda(m_1, m_2; n)/\theta(m_1, m_2; n)$ (with fixed m_1, m_2), we see that

$$f(p) = f(p^2) = \cdots = f(p^\alpha) = h(p) + 1$$

for every prime factor p of n.

Hence, from the general Brauer-Rademacher identity obtained by Subbarao [6] and Theorem 5.9, we obtain the required identity.

6. Applications to Certain Restricted Relative Partitions

We shall first prove

6.1. Lemma. *Let A be a nonempty set of positive integers and n, N be any two integers such that $0 \leq n < N$ and for a given u, where $u = 0, 1, 2, \ldots, N-1$, let*

$$C(A; u, N) \stackrel{\text{def}}{\equiv} \sum_{\substack{\ell \in A \\ r, r \equiv u \bmod N}} e_N(r\ell),$$

and we denote $C(A; 0, N)$ by $\theta(A; N)$, Then if $G(x) = \sum_{r=0}^{\infty} p_r x^r$

is a power series with a finite non zero radius of convergence, we have

$$\sum_{\ell\in A} G(e_N(\ell))e_N(-\ell n) = \theta(A;N)\Big(\sum_{t=1}^{\infty} p_{n+tN}\Big)$$

$$(6.2) \qquad + \sum_{u=1}^{N-1} C(A;u,N)\Big(\sum_{t=1}^{\infty} p_{n+tN+u}\Big)$$

This follows on collecting the terms containing r in the same residue class mod N.

Let A and B be two nonempty sets of positive integers and

$P_s(B;N,n) \stackrel{\text{def}}{\equiv}$ the number of restricted relative partitions of n modulo N for which $n \equiv \sum_{j=1}^{s} a_j \pmod N$, $a_j \in B$

and

$$P_s(A,B;N,n) \stackrel{\text{def}}{\equiv} \theta(A,N)P_s(B;N,n) + \sum_{u=1}^{N-1} C(A;u,N)P_s(B;N,n+u).$$

This $P_s(A,B;N,n)$ is a weighted relative partition function into summands belonging to B. In this, every partition of every positive integer in the residue class $0 \pmod N$ is counted $\theta(A,N)$ times and any partition of any integer belonging to any other residue class $u \pmod N$ is counted $C(A;u,N)$ times.

We then have, on utilizing a method of Subbarao [5],

6.3. Theorem. $P_s(A,B;N,n)$ *is given by*

$$P_s(A,B;N,n) = \sum_{\ell\in A} \big(C(B;\ell,n)^s \exp(-2\pi i\ell n/N)\big),$$

where

$$C(B;\ell,N)=\sum_{x\in B} e_N(\ell x).$$

Proof. We note that

$$P_s(B;N;n)=\#\{n: n\equiv a_1+a_2+\cdots+a_s \,(\mathrm{mod}\, N),\ a_j\in B\}$$

so that if

$$G(x)=\sum_{a_j\in B} x^{a_1+a_2+\cdots+a_s}=\sum_{r=0}^{\infty} p_r x^r,$$

so that

$$p_r = \text{ number of partions of } n \text{ into } s \text{ summands } \in B,$$

we have

$$P_s(B;N,u)=\sum_{r\equiv u(\mathrm{mod}\ N)} p_r$$

and substituting this in (6.2) and rewriting the left hand member of (6.2) for the present choice of $G(x)$, in terms of $C(B;\ell,n)$ the theorem follows.

6.4. Corollary. *Choosing*

$$A=\{\ell:\ell>0,\ \ell(\mathrm{mod}^* N;m_1,m_2)\}$$

$$B=\{a: a \quad runs\,(\mathrm{mod}^* N)\}$$

and by setting $P_s^*(N,n) = P_s(A,B;N,n)$ and $P_s^*(N,n+u) =$

$P_s(B; N, n)$ for these A, B we have

$$\theta(m_1, m_2; N)P_s^*(N, n) + \sum_{u=1}^{N-1} C(m_1, m_2; u, N)P_s^*(N; n+u)$$
$$= \sum_{\ell \in A} C(\ell, N)^s \exp\{-2\pi i \ell n/N\}.$$

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