ON AN EXTENSION OF NAGELL'S TOTIENT
FUNCTION AND SOME APPLICATIONS

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1. Introduction

In 1923 Nagell [4] studied the totient function \( \theta(m, n) \) for positive integers \( m, n \) defined as the number of positive integers \( x \leq n \) for which \( (x, n) = 1 = (m - x, n) \). Several interesting properties and applications of the same were studied.

We shall denote by \( \mod^*n; m \), the set of all positive integers \( x \leq n \) for which \( (x, n) = 1 = (m - x, n) \) and call it an RRS (Reduced Residue System) \( \mod n; m \). Note that \( \mod^*n; n \) is simply a reduced residue system \( \mod n \), and is denoted by \( \mod^*n \). As usual we shall denote a Complete Reduced System (CRS) \( \mod n \) by simply writing \( \mod n \). In general, we define for \( r \geq 2 \),

\[ \theta(m_1, m_2, \ldots, m_r; n) \]

as the number of positive integers \( x \leq n \) for which \( (x, n) = 1 = (m_i - x, n) \), \( i = 1, 2, \ldots, r \), where \( m_i \), \( i = 1, 2, \ldots, r \) are positive integers, and denote analogously this RRS \( \mod^*n : m_1, m_2, \ldots, m_r \). As the referee suggested, in order not to confuse this with a congruence relation, for example, say \( x = n \mod m \), we simply write \( r = n \mod m \), omitting parentheses. In this paper, we study the function \( \theta(m_1, m_2 \ldots, m_r; n) \) and obtain some of its arithmetical properties and identities that appear to be new. For the sake of simplicity we restrict ourselves mostly to the case \( r = 2 \). As an application of this function, we construct the Ramanujan Sum analogues associated with this function and study

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their properties. We obtain also some results involving an associated zeta function analogue, and obtain applications to certain restricted relative partitions mod $N$.

2. Preliminaries

Let $\mu(n)$ denote the well known Möbius function given by

$$
\mu(n) = \begin{cases} 
(-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes } (k \geq 0) \\
0 & \text{otherwise.}
\end{cases}
$$

We recall that an arithmetic function $f$ is said to be multiplicative if

$$f(mn) = f(m)f(n) \quad \text{whenever } (m,n) = 1.$$

We say that $f$ is completely multiplicative if this multiplicative property holds for all $m,n$.

Further, if

$$e_n(x) \overset{\text{def}}{=} \exp \left\{ \frac{2\pi i x}{n} \right\}$$

then

$$\sum_{d \pmod{n}} e_n(xd) = \begin{cases} 
n & \text{if } n|d; \\
0 & \text{otherwise.}
\end{cases}$$

The Ramanujan sum $C(\ell, n)$ (see [3]) and some other arithmetic functions that are needed in this paper are defined below.

$$C(\ell, n) \overset{\text{def}}{=} \sum_{d \pmod{n}} e_n(x\ell)$$

$$I_k(n) \overset{\text{def}}{=} n^k, \quad \forall n$$

$$E(n) = I_0(n) = 1, \quad \forall n$$

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\[ E_0(n) = \lfloor 1/n \rfloor = \begin{cases} 
1, & n = 1 \\
0, & n > 1. 
\end{cases} \]

\[ \mu(m_1, m_2; n) \overset{\text{def}}{=} \begin{cases} 
(-1)^{w_1(n)}(-2)^{w_2(n)}(-3)^{w_3(n)}, \\
\text{when } n \text{ is a product of distinct primes,} \\
0 & \text{otherwise.} 
\end{cases} \]

\[ \lambda(m_1, m_2; n) \overset{\text{def}}{=} 2^{\Omega_2(n)}3^{\Omega_3(n)}, \]

where

\[ w_1(n) = \# \{ \text{distinct primes } p : p \mid (n, m_1, m_2) \} \]

\[ w_2(n) = \# \{ \text{distinct primes } p : p \mid n \text{ and } p \mid \text{ just one of } m_1, m_2, m_1 - m_2 \} \]

\[ w_3(n) = \# \{ \text{distinct primes } p : p \mid n \text{ and } p \nmid \text{ any one of } m_1, m_2, m_1 - m_2 \} \]

and

\[ \Omega_i(n), i = 1, 2, 3 \text{ denote the total number of prime factors of } n \text{ corresponding to } w_i(n), i = 1, 2, 3 \text{ respectively.} \]

Let us call the primes of the above three types as primes of type 1, type 2 and type 3 respectively with respect to the pair of integers \( m_1, m_2 \).

If \( f(n), g(n) \) are any two arithmetic functions of \( n \) (where \( f \) or \( g \) or both may also be functions of some more parameters), we denote by \( \circ \) the Dirichlet Convolution of \( f \) and \( g \) with respect to \( n \), for e.g. \( I(n) \circ \mu(m_1, m_2; n) = \text{ Dirichlet product of } I(n) \text{ and } \mu(m_1, m_2; n) \text{ with respect to the argument } n. \)
3. Formulae for $\theta(m_1, m_2; n)$

We first obtain a 'Möbius type' inversion formula given by

3.1. Theorem. Let $f(x)$ be a periodic function of a real variable $x$ with period 1 and

$$F(m_1, m_2; n) = \sum_{r \equiv m_1 \pmod{n}} f(r/n),$$

then

$$F(m_1, m_2; n) = \sum \mu(d_1) \mu(d_2) G(d_1, d_2; m_1, m_2; n),$$

(3.2)

where the summation is over those divisor pairs $d_1, d_2$ of $n$ for which $(d_i, m_i) = 1$, $i = 1, 2$ and $(d_1, d_2)|(m_1 - m_2)$ and

$$G(d_1, d_2; m_1, m_2; n) = \sum_{r \equiv m_1 \pmod{n}, r \equiv m_i \pmod{d_i}, i = 1, 2} f(r/n).$$

This is easily proved by the inclusion-exclusion combinatorial principle.

In particular, choosing $f(x) = 1$, we have

3.3. Theorem. $\theta(m_1, m_2; n) = \sum \mu(d_1) \mu(d_2) \phi(n)/\phi(\delta)$ where $\delta = \text{l.c.m.} \{d_1, d_2\}$ and the summation is as in (3.2).

The proof follows easily by making use of the following well known result of Vaidyanathaswamy [7].

3.4. If $d|N$ and $(t, d) = 1$ then in any RRS $(\text{mod} N)$ there are $\phi(N)/\phi(d)$ integers congruent to $t(\text{mod} d)$.

When $f(x) = 1$, Theorem 3.1 gives

$$G(d_1, d_2; m_1, m_2; n) = \begin{cases} \frac{\phi(n)}{\phi(\delta)} & \text{if } (d_1, d_2)|(m_1 - m_2) \\ 0 & \text{otherwise} \end{cases},$$

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where $\delta = \text{l.c.m.} \{d_1, d_2\}$ by considering the simultaneous solutions of $r = m_i \pmod{d_i}$, $i = 1, 2$.

Next we note

3.5. Lemma. Let $f \neq 0$, $g, h, s$ be multiplicative arithmetic functions and for any three positive integers $n, m_1, m_2$, let

$$A = \{-\text{ordered pairs of positive integers} (d_1, d_2) \text{ such that } d_1, d_2 \mid n, \quad (d_i, m_i) = 1, \quad i = 1, 2 \text{ and } d = (d_1, d_2) \mid (m_1 - m_2)\}.$$ 

Then

$$H(m_1, m_2; n) \overset{\text{def}}{=} s(n) \sum_A g(d_1) h(d_2) / f(d_1 d_2 / d)$$

is multiplicative in $n$.

In particular, for $g = h = \mu$, $f = \phi$, $s = \mathcal{E}$, we obtain that $\theta(m_1, m_2; n)$ is multiplicative in $n$.

The result follows easily following the usual procedure of splitting each of the two divisors $d_j$ ($j = 1, 2$) of $n = n_1 n_2$ with $(n_1, n_2) = 1$ and satisfying conditions in $A$ into product of two relatively prime numbers $d_{j1}, d_{j2}$ dividing $n_1$ and $n_2$ respectively and using the multiplicativity of $f, g, h$ and $s$.

In the particular case mentioned in 3.5, we obtain through evaluation of $\theta$ at prime powers that

$$\theta(m_1, m_2; n) = \phi(n) \prod_p \left(1 - \frac{1}{\phi(p)}\right) \prod_q \left(1 - \frac{2}{\phi(q)}\right)$$

$$= n \prod_u \left(1 - \frac{1}{u}\right) \prod_p \left(1 - \frac{2}{p}\right) \prod_q \left(1 - \frac{3}{q}\right),$$

where $u, p$ and $q$ run through the prime divisors of $n$, of types 1, 2 and 3 respectively.

Note that we have

(3.6) \hspace{1cm} \theta(m_1, m_2; n) = I(n) \circ \mu(m_1, m_2; n).
Remark. More generally, it can be proved by using a similar argument that

$$\theta(m_1, m_2, \ldots, m_r; n) = \sum \mu(d_1)\mu(d_2)\cdots\mu(d_r)\phi(n)/\phi(\delta),$$

where the summation is over those \( r \)-tuples \( (d_1, d_2, \ldots, d_r) \) for which

(i) \( d_j | n \) and \( (d_j, m_j) = 1, \quad j = 1, 2, \ldots, r \)

(ii) \( (d_i, d_j) | (m_i - m_j), \quad 1 \leq i < j \leq r, \quad j = 1, 2, \ldots, r \) and

(iii) \( \delta = \{d_1, d_2, \ldots, d_r\} \), the l.c.m. of \( d_1, d_2, \ldots, d_r \).

The function \( \theta \) is multiplicative in \( n \) and so is given by

$$\theta(m_1, m_2, \ldots, m_r; n) = n \prod_{p_1} \left(1 - \frac{1}{p_1}\right) \prod_{p_2} \left(1 - \frac{2}{p_2}\right) \cdots \prod_{p_{r+1}} \left(1 - \frac{r+1}{p_{r+1}}\right),$$

wherein for \( t = 1, 2, \ldots, (r+1) \), \( p_t \) runs over those prime divisors of \( n \) which are relatively prime to just \( r + \binom{r}{2} - \left(\frac{r+2-t}{2}\right) \) of the members of the set \( M = \{m_j, (m_i - m_j) : 1 \leq i < j \leq r, \quad j = 1, 2, \ldots, r\} \), that is, those which just divide \( \left(\frac{r+2-t}{2}\right) \) members of the set \( M \).

4. Some Identities Involving \( \theta(m_1, m_2; n) \)

We define a zeta function analogue given by

$$\zeta(m_1, m_2; s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \lambda(m_1, m_2; n)/n^s$$

$$= \prod_u (1 + u^{-s} + u^{-2s} + \cdots) \prod_p (1 + 2p^{-s} + 2^2 p^{-2s} + \cdots)$$

$$\times \prod_q (1 + 3q^{-s} + 3^2 q^{-2s} + \cdots),$$
which is convergent for $\sigma = R\ell s > 1$, since

$$\left| \sum_u \sum_j u^{-js} + \sum_p \sum_j 2^j p^{-js} + \sum_q \sum_j 3^j q^{-js} \right| \leq \sum_n w(n)n^{-\sigma},$$

where $w(n) = w_1(n) + w_2(n) + w_3(n)$.

Then we have from (3.6) that

$$\sum_{n=1}^\infty \frac{\theta(m_1, m_2; n) / n^s}{n^s} = \frac{\zeta(s - 1)}{\zeta(m_1, m_2; s)}, \quad R\ell s > 2.$$  \hfill (4.1)

We further note that $\lambda(m_1, m_2; n) \circ \mu(m_1, m_2; n) = E_0(n)$ and so

$$\sum_{n=1}^\infty \frac{\mu(m_1, m_2, n)}{n^s} = \prod_u (1 - u^{-s}) \prod_p (1 - 2p^{-s}) \prod_q (1 - 3q^{-s})$$

$$= M(m_1, m_2; s), \quad \text{say},$$

which is convergent for $\sigma = R\ell s > 1$ since

$$\left| \sum u^{-s} + \sum 2p^{-s} + \sum 3q^{-s} \right| \leq 3 \sum n^{-\sigma}.$$ 

Hence we also have for $R\ell s > 1$

$$\sum_{n=1}^\infty \frac{\lambda(m_1, m_2; n) / n^s}{n^s} = 1/M(m_1, m_2; s) = \zeta(m_1, m_2; s).$$  \hfill (4.2)

Let

$$\tau(m_1, m_2; n) \overset{\text{def}}{=} E(n) \circ \lambda(m_1, m_2; n)$$  \hfill (4.3)

and

$$\sigma^{(k)}(m_1, m_2; n) \overset{\text{def}}{=} \sum_{d|n} 2^{\alpha_2(d)} 3^{\alpha_3(d)} (n/d)^k = I_k(n) \circ \lambda(m_1, m_2; n).$$  \hfill (4.4)

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Then $\tau$ is a weighted divisor function of $n$, with a weight $2^{\Omega(n/d)} 3^{\omega(n/d)}$ attached to each divisor $d$ of $n$. $\sigma^{(k)}(m_1, m_2; n)$ is the corresponding weighted sum of the $k$-th powers of divisors of $n$. We then have

\[
\sum_{1}^{\infty} \tau(m_1, m_2; n)/n^s = \zeta(s)\zeta(m_1, m_2, s), \\
\text{Re } s > 1
\]

\[
\sum_{1}^{\infty} \sigma^{(k)}(m_1, m_2; n)/n^s = \zeta(s-k)\zeta(m_1, m_2; s), \\
\text{Re } s \geq 1, \quad k \geq 1.
\]

These identities can be easily verified. We further have the following multiplicative and summatory results.

\[
\theta(m_1, m_2; n_1 n_2) = \theta(m_1, m_2; (n_1, n_2)) \\
\theta(m_1, m_2; (n_1, n_2)) = \theta(m_1, m_2; n_1) \theta(m_1, m_2; n_2)
\]

\[
\sum_{d|n} \theta(m_1, m_2; d) \mu(n/d) = \theta(m_1, m_2; n) \phi(n)/n
\]

\[
\sum_{d|n} \theta(m_1, m_2; d) \mu(m_1, m_2; n/d) = \left(\theta(m_1, m_2; n)\right)^2 /n.
\]

We shall indicate the proofs of (4.7) and (4.9).

**Proof of (4.7).** Since the functions are multiplicative, it is enough to prove the result when $n_1 = p^{a_1}$, $n_2 = p^{a_2}$ (prime powers).
If $p_1 \neq p_2$, using the multiplicativity of $\vartheta$, we have

$$\vartheta(m_1, m_2; p_1^{a_1} p_2^{a_2}) \vartheta(m_1, m_2; (p_1^{a_1}, p_2^{a_2}))$$
$$= \vartheta(m_1, m_2; p_1^{a_1}) \vartheta(m_1, m_2; p_2^{a_2}) \vartheta(m_1, m_2, 1)$$
$$= \vartheta(m_1, m_2; p_1^{a_1}) \vartheta(m_1, m_2; p_2^{a_2}) (p_1^{a_1}, p_2^{a_2}).$$

If $p_1 = p_2$ and $p_1$ divides both $m_1$ and $m_2$ we have $(p_1^{a_1}, p_2^{a_2}) = p_1^{\min(a_1, a_2)}$ and so

$$\vartheta(m_1, m_2; p_1^{a_1} p_1^{a_2}) \vartheta(m_1, m_2; (p_1^{a_1}, p_1^{a_2}))$$
$$= \vartheta(m_1, m_2; p_1^{a_1+a_2}) \vartheta(m_1, m_2; p_1^{\min(a_1, a_2)})$$
$$= p_1^{a_1+a_2} (1 - 1/p_1) p_1^{\min(a_1, a_2)} (1 - 1/p_1)$$
$$= \vartheta(m_1, m_2; p_1^{a_1}) \vartheta(m_1, m_2; p_1^{a_2}) (p_1^{a_1}, p_1^{a_2})$$
$$\text{since } p_1 = p_2.$$

The results in the other cases follow similarly.

Proof of (4.9). For $n = p^\alpha$, we have

$$\sum_{\beta=0}^{\alpha} \vartheta(m_1, m_2; p^\beta) \mu(p^{\alpha-\beta}) = \vartheta(m_1, m_2; p^\alpha) - \vartheta(m_1, m_2; p^{\alpha-1})$$
$$= \vartheta(m_1, m_2; p^\alpha)(1 - 1/p)$$
$$= \vartheta(m_1, m_2; p^\alpha) \phi(p^\alpha)/p^\alpha.$$

5. Allied Ramanujan Sum Analogues

We define two Ramanujan sum analogues:

$$C(m_1, m_2, \ell, n) \overset{\text{def}}{=} \sum_{r \equiv \ell \pmod{n/m_1, m_2}} c_n(\ell r) \quad (5.1)$$

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and

\[
C(m_1, m_2; \ell, n) \overset{\text{def}}{=} \sum_{d | (\ell, n)} d \mu(m_1, m_2; n/d).
\]

It is easy to see that \( C \) is modular or periodic in each of \( m_1, m_2 \) and \( \ell \mod n \) since \( (n, m_4 + kn - r) = (n, m_4 - r) \) and \( e((\ell + kn)r) = e_n(\ell r) \). Further, since \( ((\ell, n), n) = (\ell, n) \), \( \tilde{C} \) is an even function of \( \ell \mod n \). We refer to E. Cohen [2] for the definition and properties of even functions. Also whenever \( (n_1, n_2) = 1 \), we have \( (\ell, n_1 n_2) = (\ell, n_1)(\ell, n_2) \), where \( ((\ell, n_1), (\ell, n_2)) = 1 \) and so \( \tilde{C} \) is multiplicative in \( n \).

We also have a translation property of \( C \) given by

5.3. Theorem. If \( (m, n) = 1 \), then

\[
C(m_1, m_2; \ell m, n) = C(m_1 m, m_2 m; \ell, n).
\]

This follows on noting that when \( (m, n) = 1 \), \( r \) runs \( (\mod n; m_1, m_2) \) if and only if \( mr \) runs \( (\mod n; mm_1, mm_2) \).

In the particular case when \( n | \ell \), we have

5.3.1. Corollary. If \( (m, n) = 1 \), then

\[
\theta(m_1, m_2; n) = \theta(m_1 m, m_2 m; n).
\]

When \( n | m_1 \) and \( m_2 \) in (5.3) we have

5.3.2. Corollary. When \( (m, n) = 1 \) the Ramanujan sum \( C(\ell, n) \) satisfies \( C(\ell m, n) = C(\ell, n) \).

5.4. Theorem. Whenever \( (n_1, n_2) = 1 \), we have

\[
C(m_{11}, m_{12}; \ell_1, n_1)C(m_{21}, m_{22}; \ell_2, n_2) = C(M_1, M_2; \ell, n_1 n_2),
\]

where \( \ell = \ell_1 n_2 + \ell_2 n_1 \), \( M_1 = m_{11} n_2 + m_{21} n_1 \) and \( M_2 = m_{12} n_2 + m_{22} n_1 \).

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Proof. Let \( r_1(\mod^* n_1; m_{11}, m_{12}) \) and \( r_2(\mod^* n_2; m_{21}, m_{22}) \) and \( s = n_1 r_2 + n_2 r_1 \). Since \((n_1, n_2) = 1\) we have \((r_1, n_1) = 1\) and \((r_2, n_2) = 1\) \(\implies\) \((s, n_1 n_2) = 1\). Again because of the same reason \((m_{1i} - r_1, n_1) = 1\), \((m_{2i} - r_2, n_2) = 1\) \(\implies\) \((M_i - s, n_1 n_2) = 1\), \(i = 1, 2\).

So, also for \(s(\mod^* n_1 n_2; M_1, M_2)\) and this proves the theorem since

\[
C(m_{11}, m_{12}; \ell_1, n_1)C(m_{21}, m_{22}; \ell_2, n_2)
= \sum_{r_i(\mod^* n_i; m_{i1}, m_{i2}), i=1,2} \exp\{2\pi i(\ell_1 r_1 n_2 + \ell_2 r_2 n_1)/n_1 n_2\}
= \sum_{s(\mod^* n_1 n_2; M_1, M_2)} \exp(2\pi i s/n_1 n_2)
= C(M_1, M_2; \ell, n_1 n_2).
\]

We also have an analogue of a result of Ramanujan.

5.5. Theorem.

\[
\sigma^{(s)}(\ell) = \ell^s \zeta(m_1, m_2; s + 1) \sum_{n=1}^{\infty} \overline{C}(m_1, m_2; \ell, n)/n^{s+1}, \quad \Re s > 0,
\]

where \(\sigma^{(s)}(\ell) = \text{sum of the} \ s^{th} \ \text{powers of the positive divisors of} \ \ell.\)

This follows on noting that we can write (5.2) as \(\overline{C}(m_1, m_2; \ell, n) = I(\ell, n) \circ \mu(m_1, m_2; n)\) where

\[
I(\ell, n) = \begin{cases} n & \text{if } n | \ell \\ 0 & \text{otherwise}, \end{cases}
\]

on realizing that \(\sum_{n=1}^{\infty} I(\ell, n)/n^s = \sigma^{(1-s)}(\ell) = \sigma^{(s-1)}(\ell)/\ell^{s-1}.\)
5.6. Theorem. Hölder type identity for \( C(m_1, m_2; \ell, n) \). Whenever \( d \mid (\ell, n) \), we have

\[
C(m_1, m_2; \ell, n) = \theta(m_1, m_2; n)C(m_1, m_2; \ell/d, n/d)\theta(m_1, m_2; n/d).
\]

This follows from the definition of \( C \), using the

5.7. Lemma. For any given divisor \( d \) of \( n \), and any given \( j \) belonging to the residue system \((\text{mod}^*d; m_1, m_2)\) there are

\[
\theta(m_1, m_2; n)/\theta(m_1, m_2; d)
\]

numbers congruent to \( j \) (mod \( d \)) in the residue system \((\text{mod}^*n; m_1, m_2)\).

This lemma is easily proved on the same lines as result (3.6) above of
Vaidyanathaswamy [7].

5.8. Corollary. When \( g = (\ell, n) \),

\[
C(m_1, m_2; \ell, n) = \theta(m_1, m_2; n)C(m_1\ell/g, m_2\ell/g; 1, n/g)\theta(m_1, m_2; n/g).
\]

This follows from Lemma (5.7) and Theorem (5.3).

Next we shall obtain some identities for \( \bar{C}(m_1, m_2; \ell, n) \).

5.9. Theorem. If \( g = (\ell, n) \), then the identity

\[
\bar{C}(m_1, m_2; \ell, n) = \theta(m_1, m_2; n)\mu(m_1, m_2; n/g)/\theta(m_1, m_2; n/g)
\]

holds under the following conditions.

(i) For all \( m_1, m_2 \) when \( (n, 6) = 1 \)
(ii) For those \( m_1, m_2 \) with respect to which 2 is not of type 2, whenever \( 2 \mid n \) and
(iii) For those \( m_1, m_2 \) with respect to which 3 is not of type 3, whenever \( 3 \mid n \).

This is a particular case of the following
5.10. Theorem. Let $f$ be a completely multiplicative function and let $A(n) = \mu(m_1, m_2; n)h(n)$, where $h(n)$ is a multiplicative function. Then the sum

$$\tilde{s}(m_1, m_2; \ell, n) \overset{\text{def}}{=} \sum_{d|\ell} f(d)A(n/d), \quad g = (\ell, n)$$

satisfies the identity

$$\tilde{s}(m_1, m_2; \ell, n) = F(m_1, m_2; n)A(n/g)/F(m_1, m_2; n/g)$$

where

$$F(m_1, m_2; n) = (f \circ A)(n)$$

$$= f(n)\prod_{p|n} \left(1 + \frac{\mu(m_1, m_2; p)h(p)}{f(p)}\right)$$

provided that

(i) $f(p) \neq 0$, for all $p|n$
(ii) $f(p) \neq h(p)$ for $p|n$ of type 1
(iii) $f(p) \neq 2h(p)$ for $p|n$ of type 2 and
(iv) $f(p) \neq 3h(p)$ for $p|n$ of type 3.

(Note that $A(n)$ is actually a function of $m_1, m_2$ and $n$.) This theorem is a generalization of Theorem 8.8, pp. 163-164 of Apostol [1].

Proof. We first note that

$$\tilde{s}(m_1, m_2; \ell, n) = \sum_{d|\ell} f(d)\mu(m_1, m_2; n/d)h(n/d)$$

(noting that $n/d = (n/g)(g/d)$ has a square factor whenever
\((n/g, g/d) \neq 1\) and using the definition of \(\mu\) in the preceding step), we have with \(g/d = \delta\) that

\[
\overline{\sigma}(m_1, m_2; \ell, n) = \sum_{\delta | \ell, (\delta, n/g) = 1} f(g/\delta) \mu(m_1, m_2; \delta n/g) h(\delta n/g)
\]

\[
= f(g) \mu(m_1, m_2; n/g) h(n/g) \sum_{\delta | \ell, (\delta, n/g) = 1} \mu(m_1, m_2; \delta) h(\delta) / f(\delta)
\]

(using the complete multiplicativity of \(f\), satisfying (i) to (iv) and definition of \(\mu\))

\[
= f(g) A(n/g) \prod_{\ell | \delta, \delta | n/g} \left(1 + \frac{\mu(m_1, m_2; p) h(p)}{f(p)}\right).
\]

But

\[
F(m_1, m_2; n) = \sum_{d | n} f(d) \mu(m_1, m_2; n/d) h(n/d)
\]

\[
= f(n) \sum_{e | n} \mu(m_1, m_2; e) h(e) / f(e), \quad (e = n/d)
\]

\[
= f(n) \prod_{p | n} \left(1 + \frac{\mu(m_1, m_2; p) h(p)}{f(p)}\right),
\]

in view of multiplicativity of \(\mu\) and \(h\).

Hence we obtain the theorem using the complete multiplicativity of \(f\).

When we choose \(f(n) = n\) and \(h(n) = 1\) for all \(n\), in the above Theorem 5.9 follows.

5.11. Theorem. A Brauer-Rademacher type identity holds for \(\overline{\sigma}(m_1, m_2; \ell, n)\) under the conditions of Theorem 5.9. It is given

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\[
\theta(m_1, m_2; n) \sum_{d|m_1(\ell, d) = 1} d\mu(n/d)/\theta(m_1, m_2; d) \\
= \tilde{C}(m_1, m_2; \ell, n)\mu(n)\mu(n/g)\lambda(m_1, m_2; n/g)/\mu(m_1, m_2; n/g)
\]

where \( g = (\ell, n) \).

**Proof.** Defining \( f(n) = n/\theta(m_1, m_2; n) \), \( h(n) = \lambda(m_1, m_2; n)/\theta(m_1, m_2; n) \) (with fixed \( m_1, m_2 \)), we see that

\[
f(p) = f(p^2) = \cdots = f(p^n) = h(p) + 1
\]

for every prime factor \( p \) of \( n \).

Hence, from the general Brauer-Rademacher identity obtained by Subbarao [6] and Theorem 5.9, we obtain the required identity.

6. Applications to Certain Restricted Relative Partitions

We shall first prove

6.1. **Lemma.** Let \( A \) be a nonempty set of positive integers and \( n, N \) be any two integers such that \( 0 \leq n < N \) and for a given \( u \), where \( u = 0, 1, 2, \ldots, N - 1 \), let

\[
C(A; u, N) \overset{\text{def}}{=} \sum_{\substack{\ell \in A \\ \ell \equiv u \mod N}} e_N(r\ell),
\]

and we denote \( C(A; 0, N) \) by \( \theta(A; N) \). Then if \( G(x) = \sum_{r=0}^{\infty} p_r x^r \)
is a power series with a finite non zero radius of convergence, we have

\[
\sum_{\ell \in A} G(e_N(\ell)) e_N(-\ell n) = \theta(A; N) \left( \sum_{t=1}^{\infty} p_{n+tN} \right) + \sum_{u=1}^{N-1} C(A; u, N) \left( \sum_{t=1}^{\infty} p_{n+tN+u} \right)
\]

(6.2)

This follows on collecting the terms containing \( r \) in the same residue class \( \mod N \).

Let \( A \) and \( B \) be two nonempty sets of positive integers and

\[ P_s(B; N, n) \overset{\text{def}}{=} \text{the number of restricted relative partitions of} \ n \mod N \text{ for which} \ n = \sum_{j=1}^{s} a_j \,(\mod N), \quad a_j \in B \]

and

\[ P_s(A, B; N, n) \overset{\text{def}}{=} \theta(A, N)P_s(B; N, n) + \sum_{u=1}^{N-1} C(A; u, N)P_s(B; N, n + u). \]

This \( P_s(A, B; N, n) \) is a weighted relative partition function into summands belonging to \( B \). In this, every partition of every positive integer in the residue class \( 0 \,(\mod N) \) is counted \( \theta(A, N) \) times and any partition of any integer belonging to any other residue class \( u \,(\mod N) \) is counted \( C(A; u, N) \) times.

We then have, on utilizing a method of Subbarao [5],

6.3. Theorem. \( P_s(A, B; N, n) \) is given by

\[ P_s(A, B; N, n) = \sum_{\ell \in A} (C(B; \ell, n)^s \exp(-2\pi i \ell n/N)), \]
where

\[ C(B; \ell, N) = \sum_{\ell \in B} e_N(\ell x). \]

**Proof.** We note that

\[ P_s(B; N; n) = \# \{ n : n \equiv a_1 + a_2 + \cdots + a_s \pmod{N}, \ a_j \in B \} \]

so that if

\[ G(x) = \sum_{a_j \in B} x^{a_1 + a_2 + \cdots + a_s} = \sum_{r=0}^{\infty} p_r x^r, \]

so that

\[ p_r = \text{ number of partitions of } \ n \ \text{ into } \ s \ \text{ summands} \in B, \]

we have

\[ P_s(B; N; u) = \sum_{r \equiv u \pmod{N}} p_r \]

and substituting this in (6.2) and rewriting the left hand member of (6.2) for the present choice of \( G(x) \), in terms of \( C(B; \ell, n) \) the theorem follows.

**6.4. Corollary.** Choosing

\[ A = \{ \ell : \ell > 0, \ \ell \equiv (\mathbf{mod}^* N; m_1, m_2) \} \]

\[ B = \{ a : a \ \text{ runs} \pmod{\mathbf{mod}^* N} \} \]

and by setting \( P^*_s(N, n) = P_s(A, B; N, n) \) and \( P^*_s(N, n + u) = \)
\( P_s(B; N, n) \) for these \( A, B \) we have

\[
\theta(m_1, m_2; N)P_s^*(N, n) + \sum_{u=1}^{N-1} C(m_1, m_2; u, N)P_s^*(N; n + u) = \sum_{\ell \in A} C(\ell, N)^s \exp \left\{ -2\pi i\ell n/N \right\}.
\]

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