SOME REMARKS ON A PAPER OF RAMACHANDRA

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Abstract. We give the asymptotics of the sum $\sum_{x\leqslant n\leqslant x+h} f(n)$, where $h \gg x^{7/12+\varepsilon}$, for the multiplicative functions $f(n) = z^{\omega(n)}$, $z^{\omega(n)}|\mu(n)|$, $1/d_k(n)$, and $nd(n)$.

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1. Ramachandra [1] proved the following assertion. Let S_1, S_2, and S_3 be the sets of L-series, the derivatives, and the logarithms of L-series, respectively. $\log L(s,x)$ is defined by analytic continuation from the halfplane $\sigma = \operatorname{Re} s > 1$; for some complex z, we define

$$L(s,\chi)^z = \exp(z \log L(s,\chi)).$$

Let $P_1(s)$ be any finite power product (with complex exponents) of functions of S_1. Let $P_2(s)$ be any finite power product (with nonnegative integral exponents) of functions of S_2. Let also $P_3(s)$ denote any finite power product with nonnegative integral exponents of functions of S_3. Let c_n be a sequence of complex numbers such that $|c_n| \ll n^{\varepsilon}$ for every $\varepsilon > 0$ and

$$\sum \frac{|c_n|}{n^{\sigma}} < \infty \quad \text{for } \sigma > 1/2.$$

Let $F_0(s) = \sum_n \frac{c_n}{n^s}$. Furthermore, let

$$F_1(s) = P_1(s)P_2(s)P_3(s)F_0(s) = \sum_{n=1}^{\infty} \frac{g_n}{n^s}$$

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and

$$E(x) = \sum_{n \leqslant x} g_n.$$

Let $r (\leqslant 1/2)$ be a positive number. We define the contour $C(r)$ by starting from the circle $\{s \mid |s-1| = r\}$, removing the point $1-r$, and proceeding on the remaining portion of the circle in the anticlockwise direction. Let $C_0 = C(r)$.

Assume that r is so small that $F_1(s)$ has no singularities on the boundary and in interior of it, except, possibly, the place $s = 1$.

Let $C_1 = C(\frac{1}{\log x})$, and let L^-, L^+ be defined as the intervals on straightlines

$$L^- = \left[\left(1 - \frac{1}{r}\right) \mathrm{e}^{-i\pi},\ \left(1 - \frac{1}{\log x}\right) \mathrm{e}^{-i\pi}\right],$$

$$L^+ = \left[\left(1 - \frac{1}{\log x}\right) \mathrm{e}^{i\pi},\ \left(1 - \frac{1}{r}\right) \mathrm{e}^{i\pi}\right].$$

Let C^* be the contour going along L^- starting from $(1 - \frac{1}{r})\mathrm{e}^{-i\pi}$, then on C_1, and, finally, on L^+.

Let B be the constant occurring in the density result

$$N_\chi(\alpha, T) = \mathrm{O}\left(T^{B(1-\alpha)} (\log T)^2\right),$$

which is valid for all characters occurring in $P_1,\ P_2$, and P_3. Let $\varphi = 1 - 1/B + \varepsilon$ with arbitrary $\varepsilon > 0$.

Remark. According to Huxley's result, φ can be any constant greater than $7/12$.

THEOREM OF RAMACHANDRA L. *et x be sufficiently large and* $1 \leqslant h \leqslant x$. *Let*

$$I(x, h) = \frac{1}{2\pi i} \int_0^h \left(\int_{C_0} F_1(s)(v + x)^{s-1} \, \mathrm{d}s \right) \mathrm{d}v. \tag{1.1}$$

Then

$$E(x+h) - E(x) = I(x, h) + \mathrm{O}_\varepsilon\left(h \cdot \exp\left(-(\log x)^{1/6}\right) x^\varphi\right). \tag{1.2}$$

Ramachandra used the Hooley–Huxley contour for proving his very general theorem. Kátai [2] applied Ramachandra's theorem to obtain the uniform result

$$\sum_{\omega(n)=k,\ x \leqslant n \leqslant x+h} 1 = (1 + \sigma(1)) \frac{h (\log\log x)^{k-1}}{(k-1)! \log x},$$

uniformly for any $k \leqslant \log\log x + c_x \sqrt{\log\log x}$, where $c_x \to \infty$ sufficiently slowly, and $h \geqslant x^{\varphi+\varepsilon}$.

Sankaranarayanan and Srinivas [3] gave a version of Ramachandra's result in which the function $F_1(s)$ may depend on a parameter.

2. Assume that the conditions of Ramachandra's theorem are satisfied. Integrating on the same contour as Ramachandra did, we have

$$E(x) = J(x) + \mathrm{O}\left(x \cdot \exp\left(-(\log x)^{1/6}\right)\right), \tag{2.1}$$

where

$$J(x) = \frac{1}{2\pi i} \int\limits_{C_0} F_1(s) \frac{x^s}{s} \, \mathrm{d}s. \tag{2.2}$$

Furthermore, $I(x, h)$ can be written as

$$I(x, h) = \frac{1}{2\pi i} \int\limits_{C_0} F_1(s) \frac{(x+h)^s - x^s}{s} \, \mathrm{d}s. \tag{2.3}$$

Let

$$D(x, h, s) := \frac{1}{s} \left(\frac{(x+h)^s - x^s}{h} - x^{s-1} \right). \tag{2.4}$$

Assume that $\frac{1}{2} \leqslant |s| \leqslant 2$ and that $h = x^\eta$, $\eta < \frac{2}{3} - \frac{2r}{3}$ with small r. Then

$$\frac{(x+h)^s - x^s}{sh} = x^{s-1} + \frac{hx^{s-2}(1-s)}{2} + \mathrm{O}\left(h^3 x^{\sigma-3}\right)$$

and, thus,

$$D(x, h, s) = x^{s-1} \left(1 - \frac{1}{s}\right) + \mathrm{O}\left(h^3 \cdot x^{\sigma-3}\right),$$

which by $h^3 \cdot x^{\sigma-3} \ll x^{2-2r+r-2} \ll x^{-r}$ and $hx^{\sigma-2} \ll x^{-r}$ implies that

$$D(x, h, s) = x^{s-1} \frac{(s-1)}{s} + \mathrm{O}\left(x^{-r}\right).$$

Hence, we obtain that

$$\begin{aligned} \frac{E(x+h) - E(x)}{h} - \frac{E(x)}{x} = \frac{1}{2\pi i} \int\limits_{C_0} F_1(s) \frac{x^{s-1}}{s} (s-1) \, \mathrm{d}s \\ + \mathrm{O}\left(x^{-r}\right) + \mathrm{O}\left(\exp\left(-(\log x)^{1/6}\right)\right) \end{aligned} \tag{2.5}$$

and, thus, by (2.1) and (2.2) we have

$$\frac{E(x+h) - E(x)}{h} = \frac{1}{2\pi i} \int\limits_{C_0} F_1(s) x^{s-1} \, \mathrm{d}s + \mathrm{O}\left(\exp\left(-(\log x)^{1/6}\right)\right) + \mathrm{O}\left(x^{-r}\right). \tag{2.6}$$

Since $F_1(s)$ is analytic on the domain with boundary $C_0 \cup C^*$, we can transform the integration line on the right side of (2.6) to the contour C^*.

We have proved the following:

THEOREM 1. *Assume that $F_1(s)$ satisfies the conditions stated in Ramachandra's theorem. Let $r > 0$ and $\varepsilon > 0$ be sufficiently small constants, and let $x^{7/12+\varepsilon} \leqslant h \leqslant x^{\frac{2}{3} - \frac{2r}{3}}$. Then*

$$\frac{E(x+h) - E(x)}{h} = \frac{1}{2\pi i} \int\limits_{C^*} F_1(s) x^{s-1} \, \mathrm{d}x + \mathrm{O}\left(\exp\left(-(\log x)^{1/6}\right)\right). \tag{2.7}$$

Let us assume that

$$F_1(s) = \frac{U(s)}{(s-1)^z}, \tag{2.8}$$

where the function $U(s)$ is analytic in the disc $|s-1| \leqslant r$. Then, for each fixed k,

$$U(s) = A_0 + A_1(s-1) + \cdots + A_k(s-1)^k + (s-1)^{k+1}V(s),$$

where $V(s)$ is bounded in $|s-1| \leqslant r$.

Furthermore, since

$$\frac{1}{2\pi i}\int_{C^*} x^{s-1}(s-1)^{a-z}\,\mathrm{d}s = \frac{\Gamma(a-z)}{(\log x)^{a-z+1}}\frac{\sin\pi(a-z)}{\pi} + \mathrm{O}\left(x^{-r/2}\right) \tag{2.9}$$

(for the proof, see Lemma 8 of [7]), we deduce the following:

THEOREM 2. *Under the conditions stated above*, *we have*

$$\frac{1}{2\pi i}\int_{C^*} \frac{U(s)}{(s-1)^z}x^{s-1}\,\mathrm{d}s = \sum_{l=0}^{k} A_l \frac{\Gamma(l-z)}{(\log x)^{l-z+1}}\frac{(-1)^{l+1}\sin\pi z}{\pi} + \mathrm{O}\left(\frac{1}{(\log x)^{k+2-\mathrm{Re}\, z}}\right), \tag{2.10}$$

whenever $\mathrm{Re}\, z \leqslant k+1$.

Proof. By (2.9), we have only to prove that

$$\frac{1}{2\pi i}\int_{C^*} V(s)(s-1)^{k+1-z}\,\mathrm{d}s \tag{2.11}$$

can be majorized by the error term on the right-hand side of (2.10). The integral (2.11) extended to the contour $C(1/\log x)$ is obviously less than the error term of (2.10).

To estimate the integral on L^+ and L^-, let us write $s = 1-\tau$. Then

$$\frac{1}{2\pi}\int_{L^\pm} |V(s)|\;|(s-1)|^{k+1-\mathrm{Re}\, z}x^{-\tau}\,\mathrm{d}s \leqslant \frac{K}{2\pi}\int_{1/\log x}^{r} x^{-\tau}\tau^{k+1-\mathrm{Re}\, z}\,\mathrm{d}\tau$$
$$\ll \frac{1}{(\log x)^{k+2-\mathrm{Re}\, z}},$$

and the proof is completed.

3. Using Theorems 1 and 2, we can obtain short-interval versions of known theorems. We give some examples.

3.1. On the theorem of Sathe and Selberg. Let $\omega(m)$ be the number of distinct prime divisors of m, and let $\Omega(m)$ be the number of prime power divisors of m.

THEOREM 3. *Let* $|z| \leqslant c_1$. *Let*, *furthermore*, $x^{7/12+\varepsilon} \leqslant h \leqslant x^{2/3-\frac{2r}{3}}$, *where* r *and* ε *are arbitrary positive constants. Then*

$$h^{-1}\sum_{x\leqslant m<x+h} z^{\omega(m)} = \varphi(z)(\log x)^{z-1} + \mathrm{O}\left((\log x)^{z-2}\right);$$

here

$$\varphi(z) = \frac{1}{\Gamma(z)} \prod_p \left(1 - \frac{1}{p}\right)^z \left(1 + \frac{z}{p-1}\right).$$

Proof. Note that

$$F_1(s) = \sum \frac{z^{\omega(n)}}{n^s} = \prod_p \left(1 + \frac{z}{p^s - 1}\right),$$

and, thus, $F_1(s) = \zeta^z(s) h(s)$, where

$$(h(s, z) =) h(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^z \left(1 + \frac{z}{p^s - 1}\right),$$

and it can be expanded into an absolutely convergent Dirichlet series in $\mathrm{Re}\, s > 1/2$.

Using our Theorems 1 and 2 and Lemma 9.2 of Kubilius [4], we immediately obtain Theorem 3. Lemma 9.2 is the theorem of Selberg [5] (see also [6]).

THEOREM 3a. *Let $x^{7/12+\varepsilon} \leqslant h \leqslant x^{2/3-\frac{2r}{3}}$ with $\varepsilon, r > 0$. Let $|z| \leqslant c_1$. Then*

$$\frac{1}{h} \sum_{x \leqslant n < x+h} z^{\omega(n)} |\mu(n)| = \psi(z) (\log x)^{z-1} + \mathrm{O}\left((\log x)^{z-2}\right),$$

where

$$\psi(z) = \frac{1}{\Gamma(z)} \prod_p \left(1 - \frac{1}{p}\right)^z \left(1 + \frac{z}{p}\right).$$

Let, furthermore,

$$\prod_l ([x, x+h]) := \sum_{\substack{x \leqslant n < x+h \\ \omega(n) = l}} |\mu(n)|.$$

Then

$$\frac{1}{h} \prod_l ([x, x+h]) = \frac{6}{\pi^2} \frac{(\log\log x)^{l-1}}{(\log x)(l-1)!} \left(1 + \mathrm{O}\left(\frac{1}{(\log\log x)}\right)\right)$$

uniformly in $l \leqslant c \log\log x$ for an arbitrary constant c.

Proof. The first assertion follows from Theorem 2 by taking

$$h(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^z \left(1 + \frac{z}{p^s}\right),$$

$F_1(s) = \zeta(s)^z h(s)$, and the second one by estimating the coefficient of z^l in $\psi(z)$ (see Kubilius [4]).

3.2. Let $\Omega(n)$ be the number of prime power divisors of n, and let $|z| \leqslant 2 - \delta$ with an arbitrary positive constant δ.

THEOREM 4. *Let* $x^{7/12+\varepsilon} \leqslant h \leqslant x^{\frac{2}{3}-\frac{2r}{3}}$, *and let* $\varepsilon, r > 0$ *be arbitrary constants. Then*

$$h^{-1} \sum_{x \leqslant m \leqslant x+h} z^{\Omega(m)} = G(z)(\log x)^{z-1} + \mathrm{O}\left((\log x)^{\mathrm{Re}\, z-2}\right),$$

where

$$G(z) = \frac{1}{\Gamma(1+z)} \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z.$$

Proof. We note that

$$F_1(s) = \sum_{n=1}^{\infty} \frac{z^{\Omega(m)}}{m^s} = \prod_p \frac{1}{1 - \frac{z}{p^s}} = \zeta^z(s) h(s),$$

$$h(s) = \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - 1/p^s\right)^z,$$

and the statement immediately follows from Theorems 1 and 2.

3.3. On the sum $1/d_k(n)$ *over a short interval.* Let $d_k(n)$ be the number of solutions of the equation $n = x_1 \cdots x_k$ in positive integers $x_1, \ldots, x_k$. Then d_k is multiplicative, $d_k(p) = k$, and $d_k(p^\alpha) = \binom{k+\alpha-1}{\alpha}$.

Since

$$F_1(s) = \sum_{n=1}^{\infty} \frac{1/d_k(n)}{n^s} = \prod_p \left(1 + \frac{1}{k \cdot p^s} + \frac{1}{d_k(p^2) p^{2s}} + \cdots\right)$$

and $F_1(s)\zeta(s)^{-1/k} = h(s)$ can be expanded into a Dirichlet series which is absolutely convergent in $\mathrm{Re}\, s > 1/2$, the conditions of Ramachandra's theorem are satisfied. We can apply our Theorems 1 and 2. We have

$$U(s) = h(s)\left(\zeta(s)(s-1)\right)^{1/k}, \tag{3.1}$$

which is analytic in $|s-1| \leqslant r$ if $r < 1/2$.

THEOREM 5. *Let* $x^{7/12+\varepsilon} \leqslant h \leqslant x^{2/3-\frac{2r}{3}}$ *with* $\varepsilon, r > 0$.
Let $m \geqslant 0$ *be an arbitrary fixed positive integer*, *and let*

$$U(s) = A_0 + A_1(s-1) + \cdots + A_m(s-1)^m + (s-1)^{m+1} V(s),$$

where $U(s)$ *is defined in* (3.1). *Then*

$$\frac{1}{h} \sum_{x \leqslant n \leqslant x+h} \frac{1}{d_k(n)} = \sum_{l=0}^{m} A_l \frac{\Gamma(l-1/k)}{(\log x)^{l+1-1/k}} \frac{(-1)^{l+1} \sin \pi/k}{k} + \mathrm{O}\left(\frac{1}{(\log x)^{m+2-1/k}}\right).$$

Proof. Theorem 5 is a direct consequence of Theorems 1 and 2.

Remarks

1. A similar theorem can be proved for $\sum_{x \leqslant n \leqslant x+h} 1/g(n)$, where g is a multiplicative function, $g(p) = 1/\lambda$, and $g(n)n^\gamma \gg 1$ for some suitable constant γ.

2. Ivić [8] proved that

$$\sum_{n\leqslant x}\frac{1}{d_k(n)}=\sum_{j=0}^{N}c_{k,j}\ x(\log x)^{1/k-j}+\mathrm{O}\left(x(\log x)^{\frac{1}{k}-N-1}\right)$$

for every fixed $N>0$.

3. *On the sum* $\sum_{x\leqslant nd(n)\leqslant x+h}1$. An asymptotic formula for the number of integers n such that $nd(n)\leqslant x$ was first considered by Abbott and Subbarao [9], who proved that

$$\sum_{nd(n)\leqslant x}1\sim c\frac{x}{\sqrt{\log x}}\quad\text{for a suitable } c>0.$$

THEOREM 6. *Assume that* $x^{\frac{7}{12}+\varepsilon}\leqslant h\leqslant x^{\frac{2}{3}-\frac{2r}{3}}$ *where* ε *and* r *are small positive constants. Then*

$$\sigma(x,h):=\frac{1}{h}\sum_{x\leqslant nd(n)<x+h}1=(1+\mathrm{o}_x(1))\frac{1}{\sqrt{\log x}},\quad x\to\infty.$$

Proof. Let us write each integer n as Km, where K is square full, m is square free and $(K,m)=1$. For fixed K and l, count those integers $n=Km$ for which $\omega(m)=l$ and $nd(n)\in[x,x+h]$. Then $m\in[\frac{x}{K\cdot d(K)\cdot 2^l},\ \frac{x+h}{K\cdot d(K)\cdot 2^l}]$.

Let us note that

$$\sum_{l>l_0}\sum_{K}\frac{h}{K\cdot d(K)}\cdot\frac{1}{2^l}\ll\frac{h}{2^{l_0}}$$

and

$$\sum_{K>(\log x)^2}\frac{h}{Kd(K)}\ll\frac{h}{\log x}.$$

Let $l_0=(\frac{1}{2\log 2}+\delta)\log\log x\ (<(1-\delta)\log\log x)$. We have

$$\sigma(x,h)=\frac{1}{h}\sum_{K\leqslant(\log x)^2}\frac{1}{K\cdot d(K)}\sum_{l=0}^{l_0}T(K,l)+\frac{\mathrm{o}_x(1)}{\sqrt{\log x}},\tag{3.2}$$

where

$$T(K,l)=\#\left\{m\in\left[\frac{x}{K\cdot d(K)\cdot 2^l},\ \frac{x+h}{K\cdot d(K)\cdot 2^l}\right]\Big|\,\omega(m)=l,\ (m,K)=1,\ |\mu(m)|=1\right\}.\tag{3.3}$$

Assume that $K\leqslant(\log x)^2$. We define

$$H_K(s,z)=\sum_{(m,K)=1}\frac{z^{\omega(m)}|\mu(m)|}{m^s}=\prod_{p\nmid K}\left(1+\frac{z}{p^s}\right)$$

$$T_K(s,z)=\prod_{p|K}\frac{1}{1+\frac{z}{p^s}}=\sum_{v\in\mathcal{B}_K}\frac{\lambda(v)z^{\omega(v)}}{v^s},$$

where $\mathcal{B}_K$ is the set of integers all prime factors of which divide K.

Thus,

$$T_K(s,z) = \left(\sum_{v \in \mathcal{B}_K} \frac{\lambda(v) z^{\omega(v)}}{v^s} \right) H_1(s,z) \tag{3.4}$$

and, consequently,

$$\begin{aligned} &\#\{m \in [Y, Y+H] \mid \omega(m) = l,\ |\mu(m)| = 1\} \\ &= \sum_{\substack{v \leqslant Y+H \\ v \in \mathcal{B}_K}} \lambda(v) \# \left\{ \nu \in \left[\frac{Y}{v}, \frac{Y+H}{v} \right] \Bigm| \omega(\nu) = l - \omega(v),\ |\mu(\nu)| = 1 \right\}. \end{aligned} \tag{3.5}$$

Let us apply this relation with

$$Y = \frac{x}{K \cdot d(K) \cdot 2^l}, \qquad H = \frac{h}{K \cdot d(K) \cdot 2^l}, \quad v \leqslant (\log x)^{10}.$$

Then, by Theorem 3a,

$$\begin{aligned} &\# \left\{ \nu \in \left[\frac{Y}{v}, \frac{Y+H}{v} \right] \Bigm| \omega(\nu) = l - \omega(v),\ |\mu(\nu)| = 1 \right\} \\ &= \frac{6}{\pi^2} \frac{H}{v} \frac{(\log\log x)^{l-\omega(v)-1}}{(\log x)(l-1-\omega(v))!} \left(1 + \mathrm{O}\left(\frac{1}{\log\log x} \right) \right). \end{aligned}$$

For some fixed K and v, take the sum over $l \leqslant l_0$. Then

$$\begin{aligned} &\sum_{l=0}^{l_0} \frac{6}{\pi^2} \frac{h}{vK \cdot d(K) \log x} \frac{1}{2^{\omega(v)+1}} \frac{(\log\log x)^{l-1}}{2^{l-1}(l-1)!} \left(1 + \mathrm{O}\left(\frac{1}{\log\log x} \right) \right) \\ &= \frac{h}{K \cdot d(K) \cdot 2^{\omega(v)+1} v} \cdot \frac{1}{\sqrt{\log x}} \left(1 + \mathrm{O}\left(\frac{1}{\log\log x} \right) \right). \end{aligned}$$

Let us estimate

$$\sum_l \sum_{\substack{v \in \mathcal{B}_K \\ v > (\log x)^{10}}} \# \left\{ \nu \in \left[\frac{Y}{v}, \frac{Y+H}{v} \right] \Bigm| \omega(\nu) = l - \omega(v) \right\}.$$

This is less than

$$\frac{h}{K \cdot d(K)} \sum_{\substack{v \in \mathcal{B}_K \\ v > (\log x)^{10}}} \frac{1}{v} \ll \frac{h}{K d(K) (\log x)^v} \sum \frac{1}{\sqrt{v}}.$$

Since

$$\sum \frac{1}{\sqrt{v}} \leqslant \prod_{p \mid K} \left(1 + \frac{1}{\sqrt{p}} + \frac{1}{p} + \cdots \right) < c \cdot 2^{\omega(K)} \leqslant \log x,$$

the contribution of these terms is small.

Furthermore,

$$\sum_{v\in\mathcal{B}_L}\frac{\lambda(v)}{v\cdot 2^{\omega(v)}}=\prod_{p|K}\left(1-\frac{1}{2p}+\frac{1}{2p^2}-\cdots\right)=\prod_{p|K}\left(1-\frac{1}{2p-1}\right).$$

Summing over K up to $(\log x)^2$, we get

$$\sum_{K<(\log x)}\frac{1}{2K\cdot d(K)}\prod_{p|K}\left(1-\frac{1}{2p-1}\right)=c+\mathrm{O}\left(\frac{1}{\sqrt{\log x}}\right).$$

Hence, the theorem follows.

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