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A NOTE ON THE ARITHMETIC FUNCTIONS $C(n, r)$ AND $C^*(n, r)$

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1. *Preliminaries*. Throughout this note we write $e(a, b)$ for $\exp(2\pi ia/b)$. The well-known arithmetic function $C(n, r)$ of RAMANUJAN [4] is defined by

$$C(n, r) = \sum_{\substack{t (\mathrm{mod}\, r) \\ (t, r)=1}} e(nt, r).$$

(Following current practice, we write $C(n, r)$ in the place of Ramanujan's $C_r(n)$). RAMANUJAN showed ([4], equation (18)) that

$$C(1, r)/1 + C(2, r)/2 + \ldots = -\Lambda(r), \qquad (r > 1), \tag{1.2}$$

where $\Lambda(r)$ is the classical arithmetic function which equals $\log p$ if $r > 1$ is the power of a prime p, and is zero otherwise. His proof, however, is long and round about. The formula (1.2) is obtained as the limiting case, when $s \to 1$, of the result

$$\zeta(s) \sum_{\delta\delta'=r} \mu(\delta)\delta'^{(1-s)} = C(1, r)/1^s + C(2, r)/2^s + \ldots, \qquad (s > 0),$$

which itself is derived from the expansion ([4], (7.2)):

$$\sigma_{-s}(n)/\zeta(s+1) = C(n, 1)/1^{s+1} + C(n, 2)/2^{s+1} + \ldots, \qquad (s > 0);$$

The last one is got as a consequence of some general considerations. Here $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n, and $\zeta(s)$ is the Riemann zeta function.

Actually, however, (1.2) follows very easily from the well-known elementary result (usually attributed to KRONECKER) that

$$\prod_{\substack{t (\mathrm{mod}\, r) \\ (t, r)=1}} (1 - e(t, r)) = \exp \Lambda(r), \qquad (r > 1),$$

on taking logarithms of both sides and then suitably grouping the terms in the series expansions in powers of $e(t, r)$.

Following similar methods we will obtain in this note two results which are believed to be new. The first (Theorem 1) is a generalization of (1.2) and involves Cohen's generalization $C_k(n, r)$ [1] of $C(n, r)$. The second (Theorem 2) is the unitary analogue of (1.2) satisfied by the trigonometric sum $C^*(n, r)$ [3].

2. *Definitions and lemmas.* For convenience of later reference, we recall the definitions of $C_k(n, r)$ and $C^*(n, r)$ and some results to be used later.

Let $(a, b)_k$ denote the greatest kth power divisor common to a and b. Then $C_k(n, k)$ is defined by

$$C_k(n, r) = \sum_{\substack{t(\mathrm{mod}\, r^k) \\ (t, r^k)_k = 1}} e(nt, r^k).$$

We call a and b to be relatively k-prime if $(a, b)_k = 1$. The set of all $t(\mathrm{mod}\, r^k)$ which are relatively k-prime to r^k will be called a k-reduced residue system (mod r). We have

Lemma 1 (Eckford Cohen [2], lemma 4). *The numbers a,*

$$a = X(r/d)^k, \quad d \mid r, \quad (X, d^k)_k = 1,$$

where d ranges over the positive divisors of r, and for each d, X ranges over a k-reduced residue system (mod d), *form a complete residue system* (mod r^k).

We next define $C^*(n, r)$, the unitary analogue of $C(n, r)$. A divisor d of r is called 'unitary' whenever $(d, r/d) = 1$. We write $d \| r$ to indicate that d is a unitary divisor of r. For integers, $a, b, b > 0$, we write $(a, b)_*$ for the greatest divisor of a which is a unitary divisor of b, and call it the unitary g.c.d. of a with b. If $(a, b)_* = 1$, a is said to be semiprime to b. The set of integers in a complete residue system (mod r) which are semiprime to r is designated the semi-reduced residue system (mod r). We note

Lemma 2 (Eckford Cohen [3], lemma 2.1). *The integers dx, where d ranges over the unitary divisors of r, and for each d, x ranges over a semi-reduced residue system* (mod r/d), *constitute a residue system* (mod r).

The function $C^*(n, r)$ is defined by

$$C^*(n, r) = \sum_{\substack{t(\mathrm{mod}\, r) \\ (t, r)_* = 1}} e(nt, r).$$

We define

$$\mu^*(r) = C^*(1, r),$$

or, equivalently, by

$$\mu^*(r) = (-1)^{w(r)}$$

where $w(r)$ denotes the number of distinct prime divisors of r, with

$$w(1) = 0 \quad ([3], \text{theorem } 2.5).$$

The unitary analogue ([3], theorem 2.3) of the Möbius inversion formula (product form) says that for arithmetical functions $f_1(r)$ and $f_2(r)$,

Lemma 3 $f_1(r) = \prod_{d \| r} f_2(d) \Leftrightarrow f_2(r) = \prod_{d \| r} (f_1(r/d))^{\mu^*(d)}$.

3. We shall prove

Theorem 1. *For* $r > 1$,

$$C_k(1, r)/1 + C_k(2, r)/2 + C_k(3, r)/3 + \dots = -k\Lambda(r). \qquad (3.1)$$

Theorem 2. *For* $r > 1$,

$$C^*(1, r)/1 + C^*(2, r)/2 + C^*(3, r)/3 + \dots = -\Lambda^*(r), \qquad (3.2)$$

where $\Lambda^*(r)$ *is defined to be* $\log r$ *or* 0 *according as* r *is a prime power or not.*

Proof of Theorem 2. Define

$$f(r) = \prod_{t=1}^{r-1} (1 - e(t, r)), \quad f(1) = 1; \qquad (3.3)$$

and

$$g(r) = \prod_{\substack{t (\mathrm{mod}\, r) \\ (t, r)_* = 1}} (1 - e(t, r)), \quad g(1) = 1. \qquad (3.4)$$

Grouping the factors in the product for $f(r)$ according to the unitary g.c.d. of t with r and using lemma 2, we have

$$f(r) = \prod_{d \| r} g(d). \qquad (3.5)$$

An application of lemma 3 gives

$$g(r) = \prod_{d \| r} (f(r/d))^{\mu^*(d)} = \prod_{d \| r} (r/d)^{\mu^*(d)}. \qquad (3.6)$$

Let now $r > 1$ and have the canonical form $r = p_1^{a_1} p_2^{a_2} \dots p_s^{a_s}$. Then the unitary divisors of r are the products formed by i factors taken from $p_1^{a_1}, \dots, p_s^{a_s}$ $(i = 1, 2, \dots, s)$. Let t_i denote the product of all those unitary divisors of r which are products formed by exactly i of the numbers $p_1^{a_1}, \dots, p_s^{a_s}$, for a fixed value of i, $0 \leq i \leq s$. With the convention that empty products have the value unity, we have for $r > 1$, on using (3.6),

$$g(r) = t_s\, t_{s-1}^{-1}\, t_{s-2}\, t_{s-3}^{-1} \cdots$$
$$= \begin{cases} r & \text{for } s = 1 \\ 1 & \text{otherwise} \end{cases}$$
$$= \exp(\Lambda^*(r)).$$

Hence, if $r > 1$,

$$\Lambda^*(r) = \log g(r)$$
$$= -\sum_{\substack{1 \le t \le r-1 \\ (t,r)_* = 1}} \sum_{s=1}^{\infty} [e(ts, r)]/s$$
$$= -\sum_{s=1}^{\infty} C^*(s, r)/s,$$

thus completing the proof of Theorem 2.

The proof of Theorem 1 is on similar lines. We group the factors in the product for $f(r^k)$ according to the value of $(t, r^k)_k$ using Lemma 1. We get $r^k = f(r^k) = \prod_{d/r} h(d)$, where

$$h(r) = \prod_{\substack{t (\mathrm{mod}\, r^k) \\ (t, r^k)_k = 1}} (1 - e(t, r^k)).$$

Hence

$$h(r) = \prod_{d/r} (r/d)^{k\mu(d)}$$
$$= k\Lambda(r).$$

For the rest of the proof, we proceed exactly as before.

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