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ODD PERFECT NUMBERS: SOME NEW ISSUES

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1. Introduction

In this paper, the letters a, b, m, n, k, r, n, v, denote positive integers and p denotes an odd prime. As usual, $\sigma(n)$ denotes the sum of the positive divisors of n. As is well known, if $\sigma(n) = 2n$, then n is said to be a perfect number. The determination of all the perfect numbers is probably the oldest unsolved problem in mathematics challenging both human and computer capability. The classical Euler result characterizing all even perfect numbers n, namely that they are given by

$$n = 2^r(2^{r+1} - 1), \tag{1.1}$$

provided $2^{r+1} - 1 = 2^p - 1$ is a prime, (so called Mersenne prime) is helping find more and more even perfect numbers.

The latest — the 37^{th} even perfect number, announced in January 1998, is the one corresponding to $r = 3021376$, and Professor John Brillhart confirmed this (October 5, 1998); but further details are lacking, the previous one being that corresponding to $r = 2976220$, discovered by Gordon Spence of England (Mathematical Association of America Newsletter FOCUS, Vol. 17, December issue of 1997).

The most baffling unsolved problem of course concerns the existence or nonexistence of an odd perfect number. For an earlier account of known results on this we refer to McCarthy [8]. Among recent results worthy of notice we may mention that there is no odd perfect number with less than 301 digits (Brent [1]), or with less than eight distinct prime divisors (Hagis [6], Chein [2]). Also, if an odd perfect number $n$has at most r distinct prime divisors, then $n < 4^{4^r}$, (Heath-Brown [7], improving an earlier result of Pomerance [9]). Two hundred years back, Euler proved that every odd perfect number n must satisfy the requirements below:

$$n = p^a k^2, \quad p \equiv a \equiv 1 \pmod 4. \tag{1.2}$$

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For an improvement of Euler's result we refer to Ewell ([5]). For an integer n given by (1.1), if we have

$$\sigma(k^2) = p^a; \quad \sigma(p^a) = 2k^2, \tag{1.3}$$

then, of course, n is odd perfect. But should every odd perfect number n — which necessarily has the form (1.2) — also necessarily satisfy the relations (1.3)?

This question raised by Suryanarayana [11] was answered in the negative by D. G. Dandapat, J. L. Hunsucker and C. Pomerance [4], and also later by E.Z. Chen utilizing a deep result of Ljunggern. We refer to [11] for details.

Suryanarayana [11] then raised the following further

(1.4) PROBLEM. If $n = p^a k^2$ is an odd perfect number, so that

$$p \equiv a \equiv 1 \pmod 4 \quad \text{and} \quad (p, k) = 1,$$

does it necessarily follow that there exists a divisor d of k such that $\sigma(d^2) = p^a k^2/d^2$, and $\sigma(p^a k^2/d^2) = 2d^2$?

As far as this author knows, this problem is still open.

Further, Suryanarayana gave an interesting new twist to the problem of the existence of odd perfect numbers by his observation [11] that every even perfect number is expressible in the form $n = m\sigma(m)$ with $m = 2^r$, $2^{r+1} - 1$ prime, so that $(m, \sigma(m)) = 1$ and $\sigma(\sigma(m)) = 2m$. He then asked the following (we combine his Problems 1 and 2):

(1.5) PROBLEM. Is it true that every odd perfect number is of the form $m\sigma(m)$ for some odd integer m (which necessarily must be a square since otherwise $\sigma(m)$ would be even); if so, is $(m, \sigma(m)) = 1$ necessarily?

This problem also is still open.

The purpose of this note is to give a further new twist to the Euler result on even perfect numbers and raise some new problems concerning odd perfect numbers and obtain some new results including two interesting characterizations of even perfect numbers (see Theorems 2.9 and 2.10 below).

2. Some new problems

We begin by noting that every even perfect number n characterized by (1.1) can also be expressed in the form

$$n = \frac{1}{2}\, m\sigma(m), \tag{2.1}$$

and satisfies the following obvious properties:

$$m \quad \text{is prime} \tag{2.2}$$

$$\text{g.c.d. } (m, \sigma(m)) = 1 \tag{2.3}$$

$$\sigma(m) \quad \text{is a power of} \quad 2 \tag{2.4}$$

(2.5) $$\sigma(\sigma(m)) = 2m + 1.$$

These, of course, are trivial observations, but they lead to some questions concerning odd perfect numbers.

(2.6) PROBLEM. Does every odd perfect number n (if such exist) have the representation (2.1)?

(2.7) PROBLEM. Given (2.5) alone, is the number n given by (1.1) perfect?

(2.8) PROBLEM. Are the solutions (2.5) all given by Mersenne prime values of m?

We are unable to settle these questions. However, we show the following:

(2.9) THEOREM. *The relations* (2.1), (2.3), (2.5) *characterize all even perfect numbers; thus there is no odd perfect number of the form* (2.1) *satisfying* (2.3) *and* (2.5).

(2.10) THEOREM. *Given* (2.4) *and* (2.5), *they imply that an n given by* (2.1) *is even perfect; the converse is also true. Thus no odd perfect number is of the form n given by* (2.1) *satisfying* (2.4) *and* (2.5).

The result of Theorem 2.10 is contained in the following:

(2.11) THEOREM. *If* $\sigma(\sigma(m)) = 2m + 1$, *then*

(2.12) *m must be odd;*

(2.13) *$\sigma(m)$ must be of the form $2^r M^2$ where M is an odd integer, and if $M = 1$, then m is a Mersenne prime;*

(2.14) *if $\sigma(\sigma(p)) = 2p + 1$ for a prime p, then either p is a Mersenne prime or M in* (2.7) *should contain more than one prime divisors.*

3. Proofs of the theorems

PROOF OF THEOREM 2.9. Given (2.3) and (2.5) we first show that m must be odd.

Case 1. $m = 2t$ (t odd). Then, using (2.3), and $\sigma(m) = 3\sigma(t)$, we see that if n given by (2.1) is perfect, then we have

$$m\sigma(m) = \sigma\left(\frac{1}{2}\, m\sigma(m)\right) = \sigma\left(\frac{1}{2}\, m\right) \cdot \sigma(\sigma(m)) = \frac{\sigma(m)}{3} \cdot (2m + 1).$$

Hence $(2m + 1)/3 = m$, which is impossible.

Case 2. Let $m = 2^\alpha t \quad (\alpha > 1)$. Then $\sigma(m/2) = (2^\alpha - 1)\sigma(t)$; and so if n is perfect we have

$$m\sigma(m) = \sigma\left(\frac{1}{2}\, m\sigma(m)\right) = \sigma\left(\frac{1}{2}\, m\right) \cdot \sigma\big(\sigma(m)\big)$$
$$= (2^\alpha - 1)\sigma(t)(2m + 1).$$

Using $m\sigma(m) = 2^\alpha t\sigma(t)(2^{\alpha+1} - 1)$, after a routine simplification, the above equation gives

$$(2^\alpha - 1)(2m + 1) = 2^\alpha \cdot (2^{\alpha+1} - 1),$$

which is impossible, because the left side is odd while the right side is even, noting that $\alpha > 0$.

Thus from the above two cases, we conclude that m must be odd.

Now $\sigma(m)$ must be even (in order that n given by (2.1) may be perfect), for otherwise $\frac{1}{2}\, m\sigma(m)$ will not be an integer.

Case 3. m odd, $\sigma(m) = 2(2t + 1)$. We now have

$$\sigma\big(\sigma(m)\big) = 3\sigma(2t + 1) = 2m + 1$$

on using (2.5). Hence

$$\sigma\left(\frac{1}{2}\, m\sigma(m)\right) = \sigma\big(m(2t + 1)\big) = \sigma(m) \cdot \sigma(2t + 1)$$
$$= 2(2t + 1)\sigma(2t + 1) = 2(2t + 1)(2m + 1)/3$$

on using (2.5) which gives $2m + 1 = \sigma\big(\sigma(m)\big) = 3\sigma(2t + 1)$. If $n = \frac{1}{2}\, m\sigma(m)$ is even, we should then have

$$2(2t + 1)(2m + 1)/3 = m\sigma(m)$$
$$= m2(2t + 1).$$

This gives $m = (2m + 1)/3$, which is absurd. Thus this case is impossible if n is perfect.

Case 4. $\sigma(m) = 2^\alpha(2t + 1)$, $\alpha > 1$. Since, $\big(m, \sigma(m)\big) = 1$ by (1.3), we have m odd. Also m is prime and $\frac{1}{2}\, m\sigma(m)$ is even perfect number as seen from the fact that $\frac{1}{2}\, m\sigma(m) = \frac{1}{2}\, m2^\alpha(2t + 1)$, $\alpha > 1$, and this is perfect only if $m(2t + 1)$ is prime, that is, only if $t = 0$ and m is a prime.

Proof of Theorem 2.11. Suppose

$$\sigma\big(\sigma(m)\big) = 2m + 1 \quad \text{where} \quad m = 2^r N,\ r > 0,\ N \quad \text{odd}.$$

If $N > 1$, and if $\sigma(N) \neq 2^{r+1} - 1$,

$$\sigma(m) = \sigma(2^r N) = (2^{r+1} - 1)\sigma(N)$$
$$\sigma\big(\sigma(m)\big) = \sigma\big((2^{r+1} - 1)\sigma(N)\big)$$
$$\geq 1 + \sigma(N) + (2^{r+1} - 1)\sigma(N)$$
$$= 2^{r+1}\sigma(N) + 1 > 2^{r+1}N + 1$$

$$= 2m + 1$$

so that

$$\sigma(m) > 2m + 1.$$

If $\sigma(N) = 2^{r+1} - 1$, then

$$\begin{aligned}
\sigma\big(\sigma(m)\big) &= \sigma\big((2^{r+1} - 1)^2\big) \\
&\geq 1 + (2^{r+1} - 1) + (2^{r+1} - 1)^2 \\
&= 2^{r+1} + (2^{r+1} - 1)^2 \\
&= 2^{r+1} + 2^{2r+2} - 2 \cdot 2^{r+1} + 1 \\
&= 2^{2r+2} - 2^{r+1} + 1 \\
&> 2 \cdot 2^r(2^{r+1} - 3) + 1 \geq 2m + 1,
\end{aligned}$$

noting that $r \geq 1$ and $\sigma(N) = 2^{r+1} - 1$ so that, N being odd, we have

$$N \leq 2^{r+1} = 3.$$

Thus $\sigma\big(\sigma(m)\big) > 2m + 1$ in this case also. If $N = 1$,

$$\sigma\big(\sigma(m)\big) = \sigma(2^{r+1} - 1) = 2^{r+1}, \text{ if } \quad 2^{r+1} - 1 \quad \text{is a prime.}$$

Hence in this case, $\sigma\big(\sigma(m)\big) = 2m \neq 2m + 1$. If $2^{r+1} - 1$ is not a prime, let d be one of its proper divisors, so that $1 < d < 2^{r+1} - 1$. Then

$$\begin{aligned}
\sigma\big(\sigma(m)\big) &= -\sigma(2^{r+1} - 1) \geq 2^{r+1} - 1 + 1 + d \\
&= 2^{r+1} + d \\
&> 2m + 1.
\end{aligned}$$

Hence m cannot be even and (2.12) follows.

To prove (2.13), if $\sigma(m) = 2^r N$, N odd, since $\sigma\big(\sigma(m)\big) = 2m + 1 =$ an odd integer, it follows that $\sigma(N)$ should be odd, that is, N must be a perfect square. Let us suppose that $N = M^2$, M odd. If $M = 1$,

$$\sigma(m) = 2^r M^2 = 2^r$$

$$\sigma\big(\sigma(m)\big) = 2^{r+1} - 1 = 2m + 1,$$

that is, $m = 2^r - 1$, and $\sigma(m) = 2^r$. Hence m is a Mersenne prime. This proves (2.13). Finally, suppose

$$\sigma\big(\sigma(p)\big) = 2p + 1.$$

Then

$$\sigma(p) = 2^r M^2.$$

If $M = 1$, we have already seen that p must be a Mersenne prime.

If $M > 1$, suppose M contains only one prime factor, say $M = q^b$. Then

$$\sigma(p) = 2^r q^{2b}$$

and

$$\sigma\big(\sigma(p)\big) = (2^{r+1} - 1)\Big(\frac{q^{2b+1} - 1}{q - 1}\Big) = 2p + 1.$$

Hence $p+1 = 2^r q^{2b}$, $2p+1 = (2^{r+1}-1)\left(\frac{q^{2b+1}-1}{q-1}\right)$, so that

$$2^{r+1}q^{2b} - 1 = (2^{r+1}-1)\left(q^{2b} + \frac{q^{2b}-1}{q-1}\right).$$

Thus

$$(2^{r+1}-1)q^{2b} + q^{2b} - 1 = (2^{r+1}-1)q^{2b} + (2^{r+1}-1)\ \frac{q^{2b}-1}{q-1}\ .$$

It follows that

$$2^{r+1} - 1 = q - 1, \text{ so that } \quad q = 2^{r+1}.$$

This contradicts that q is odd. Hence M should have more than one distinct prime divisor.

4. Some remarks

Clearly n would not be perfect if we are given only (2.2) and (2.3), or only (2.3) and (2.4). Also if we are given (2.2) and (2.4) only, they may not imply that n is perfect, as indicated by Theorem 2.10.

Finally, in connection with Problem 2.6 stated earlier we may ask: whenever n given by (2.1) is perfect, does it follow that m is odd and $\big(m, \sigma(m)\big) = 1$?

We checked that the only solution of $\sigma\big(\sigma(m)\big) = 2m+1$ for $m \le 250000$ are the first six Mersenne primes 3, 7, 31, 127, 8191, 131071. One may probably conjecture that Mersenne primes are the only solutions of this equation, but this may not be easy to settle.

It may interest the reader to know that there exist infinitely many odd almost perfect numbers n, in the sense that $\sigma(n)/n$ differs from 2 by an amount less than ε, where $\varepsilon > 0$ is an arbitrarily small but fixed number (see S. G. Cramer [3]). For an extension of the perfect numbers concept to complex numbers, see Spira [10].

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