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SOME ROGERS-RAMANUJAN TYPE PARTITION THEOREMS

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Analogous to the celebrated Rogers-Ramanujan partition theorems, we obtain four partition theorems wherein the minimal difference for 'about the first half' of the parts of a partition (arranged in non-increasing order of magnitude) is 2. For example, we prove that the number of partitions of n, such that the minimal difference of the 'first half of the summands' (that is, first $[(t+1)/2]$ summands in a partition into t summands) of any partition is 2, equals the number of partitions n into summands congruent to $\pm 1, \pm 2, \pm 5, \pm 6, \pm 8, \pm 9 \pmod{20}$.

1. Introduction. Throughout this paper, $|x| < 1$ and we use the notation:

$$(x)_n = (1-x)(1-x^2)\cdots(1-x^n), \qquad n = 1,2,\ldots;$$
$$\phi(a,x) = (1-a)(1-ax)(1-ax^2)\cdots \quad \text{to } \infty.$$
$$\phi(x) = \phi(x,x).$$

$\pi_t(n)$ denotes a partition of n into t parts arranged in non-increasing order of magnitude, say,

(1.1) $$\pi_t(n) = n_1 + n_2 + \cdots + n_t; \qquad n_1 \geq n_2 \geq \cdots \geq n_t.$$

For convenience, we shall refer to $n_1, n_2, \ldots$ as the first, second,... part in $\pi_t(n)$. The differences $n_1 - n_2, n_2 - n_3, \ldots$ are referred to as the first, second,... differences of the parts of $\pi_t(n)$. The "first half of the parts in $\pi_t(n)$" are defined to be the parts $n_1, n_2, \ldots, n_{[(t+1)/2]}$, where $[x]$ denotes the greatest integer function. Thus these parts are

$$n_1, n_2, \ldots, n_{t/2} \quad \text{if } t \text{ is even},$$

and

$$n_1, n_2, \ldots, n_{(t+1)/2} \quad \text{if } t \text{ is odd}.$$

These are also described as the parts in the first half of the partition. We shall also have occasion to speak frequently about the minimal differences of the first half of the parts in $\pi_t(n)$. This will be the minimum of the differences

$$n_1 - n_2, n_2 - n_3, \ldots, n_{[(t-1)/2]} - n_{[(t+1)/2]},$$

the last of these being "the last difference in the first half of the summands of $\pi_t(n)$".

The celebrated Rogers-Ramanujan identities have a well-known partition-theoretic interpretation first noticed by MacMahon and can be stated thus ([**1**], Theorems 364, 365):

(1.2) The number of partitions of n into parts with minimal difference 2 equals the number of partitions of n into parts which are $\equiv \pm 1 \pmod 5$.

(1.3) The number of partitions of n with minimal part 2 and minimal difference 2 equals the number of partitions of n into parts which are $\equiv \pm 2 \pmod 5$.

In this paper, we obtain analogous theorems for partitions wherein the minimal difference of the parts in the *first half* of the partition is 2 (with one possible exception in the cases of some theorems).

For establishing our theorems, we utilize some identities of Slater [**3**]. Four of these identities were earlier used by Hirschhorn [**2**] to establish entirely different combinatorial results. However the partition theorems that we obtain here from these four identities are much more analogous in structure to the Rogers-Ramanujan partitions than those obtained by Hirschhorn.

2. The partition theorems.

2.1. THEOREM. *The number of partitions of n such that the parts in the first half of each partition have minimal difference* 2 *is equal to the number of partitions of n into parts which are congruent to* $\pm 1, \pm 2, \pm 5, \pm 6, \pm 8, \pm 9 \pmod{20}$.

2.2. THEOREM. *The number of partitions of n such that the parts in the first half of each partition have minimal difference* 2—*with the possible exception of the last difference which is at least* 1—*is equal to the number of partitions of n into parts which are congruent to* $\pm 1, \pm 3, \pm 4, \pm 5, \pm 7, \pm 9 \pmod{20}$.

2.3. THEOREM. *The number of partitions of n into parts not less than* 2 *and such that in any partition into t parts, the first* $[t/2]$ *parts have minimal size* $[t/2] + 1$ *or* $[t/2] + 3$ *according as t is even or odd and minimal difference* 2, *equals the number of partitions of n into parts which are congruent to* $\pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}$. *Equivalently, the number of partitions of n with* $n = a_1 + \cdots + a_{2s-1}$, *where* $a_1 - a_2 \geq 2, \ldots, a_{s-2} - a_{s-1} \geq 2$, $a_{s-1} \geq s + 2$, $a_s \geq a_{s+1} \geq \cdots \geq a_{2s-1} \geq 2$, *or with* $n = a_1 + \cdots + a_{2s}$ *with* $a_1 - a_2 \geq 2, \ldots, a_{s-1} - a_s \geq 2$, $a_s \geq s + 1$, $a_{s+1} \geq a_{s+2} \geq \cdots \geq a_{2s} \geq 2$ = *number of partitions of n with parts congruent to* $\pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}$.

2.4. THEOREM. *The number of partitions of n into parts not less than* 2 *and such that for the first* $[t/2]$ *parts of a partition of n into t parts, the minimum difference is* 2 *and minimum part* $[(t+1)/2]+1$, *and further, if t is odd, the middle part* (*i.e. the* $(t+1)/2$-*th*) *is at least* $(t+1)/2$, *equals the numbers of partitions of n into parts congruent to* $\pm 1, \pm 4, \pm 6, \pm 7$ (mod 16).

3. Proofs of the theorems.

3.1. *Proof of Theorem* 2.1. Consider a partition $\pi_t(n)$, satisfying the conditions on the differences of parts stated in the first part of the theorem. Let

$$\pi_t(n) = n_1 + n_2 + \cdots + n_t,$$

the parts being in non-increasing order of magnitude.

Case 1. t even $= 2s$. Then we have

$$1 \leq n_{2s} \leq n_{2s-1} \leq n_{2s-2} \leq \cdots \leq n_{s+1} \leq n_s;$$

further

$$n_{s-1} \geq n_s + 2, \quad n_{s-2} \geq n_{s-1} + 2, \quad n_2 \geq n_3 + 2, \quad n_1 \geq n_2 + 3.$$

Hence

$$n_s \geq 1, \quad n_{s-1} \geq 3, \quad n_{s-2} \geq 5, \ldots, n_1 \geq 2s-1.$$

It follows that

$$n_1 + n_2 + \cdots + n_s \geq ((2s-1) + (2s-3) + \cdots + 1) = s^2$$

and

$$n_{s+1} + n_{s+2} + \cdots + n_{2s} \geq 1 + 1 + \cdots + 1 = s.$$

Thus

$$n - (s^2+s) = (n_1 - (2s-1)) + (n_2 - (2s-3)) + \cdots + (n_s - 1)$$
$$+ (n_{s+1} - 1) + (n_{s+2} - 1) + \cdots + (n_{2s} - 1),$$

which represents a partition of $n - (s^2+s)$ into at most $2s$ parts. Thus the partitions of the type $\pi_{2s}(n)$ with the stated restrictions on differences of parts are generated by $x^{s^2+s}/(x)_{2s}$ $(s = 1, 2, \ldots)$.

Case 2. t odd $= 2s-1$. A typical partition $\pi_{2s-1}(n)$ is of the form

$$n = n_1 + n_2 + \cdots + n_{2s-1},$$

with

$$n_1 \geq n_2 + 2, \qquad n_2 \geq n_3 + 2,$$
$$\dots$$
$$n_{s-1} \geq n_s + 2, \qquad n_s \geq n_{s+1}, \qquad n_{s+1} \geq n_{s+2},$$
$$\dots$$
$$n_{2s-2} \geq n_{2s-1}, \qquad n_{2s-1} \geq 1.$$

Hence we have that each of

$$n_{s+1}, n_{s+2}, \dots, n_{2s-1} \text{ is } \geq 1,$$

while

$$n_s \geq 1, \quad n_{s-1} \geq 3, \quad n_{s-2} \geq 5, \dots, n_2 \geq 2s - 3, \quad n_1 \geq 2s - 1.$$

Hence

$$\begin{aligned} n &- (s^2 + s - 1) \\ &= (n_1 + \cdots + n_{2s-1}) - ((2s-1) + (2s-3) + \cdots + 1) - s \\ &= (n_1 - (2s-1)) + (n_2 - (2s-3)) + \cdots + (n_s - 1) \\ &\quad + (n_{s+1} - 1) + \cdots + (n_{2s-1} - 1), \end{aligned}$$

and this represents a partition of $n - (s^2 + s - 1)$ into at most $2s - 1$ parts. Thus the partitions of the type $\pi_{2s-1}(n)$ with the stated restrictions on the parts are generated by $x^{s^2+s-+1}/(x)_{2s-1}$ $(s = 1, 2, \dots)$. Noting that

$$1 + \sum_{s=1}^{\infty} \left\{ \frac{x^{s^2+s-1}}{(x)_{2s-1}} + \frac{x^{s^2+s}}{(x)_{2s}} \right\} = \sum_{s=0}^{\infty} \frac{x^{s^2+s}}{(x)_{2s+1}},$$

we get Theorem 2.1 in view of the following identity of L. J. Slater ([**3**], 94):

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{x^{r^2+r}}{(x)_{2r+1}} = \{ &\phi(x, x^{20})\phi(x^2, x^{20}) \cdot \phi(x^5, x^{20}) \cdot \phi(x^6, x^{20}) \\ &\cdot \phi(x^8, x^{20}) \cdot \phi(x^9, x^{20}) \cdot \phi(x^{11}, x^{20}) \cdot \phi(x^{12}, x^{20}) \\ &\cdot \phi(x^{14}, x^{20}) \cdot \phi(x^{15}, x^{20}) \cdot \phi(x^{18}, x^{20}) \cdot \phi(x^{19}, x^{20}) \}^{-1}. \end{aligned}$$

3.2. *Indication of the proof of the other theorems.* As suggested by the referee, we omit proofs of the other theorems, since they are analogous to that of Theorem 2.1. For Theorems 2.2, 2.3, 2.4, we utilize, respectively, the identities 79, 98, 39, 83 and 38, 86 of Slater [**3**]. We only observe that if

$$a_1 - a_2 \geq 2, \dots, a_{s-2} - a_{s-1} \geq 2, \qquad a_{s-1} \geq s + 2,$$
$$a_s \geq a_{s+1} \geq \cdots \geq a_{2s-1} \geq 2,$$

then

$$a_1 + \cdots + a_{2s-1} \geq 2s^2;$$

while if

$$a_1 - a_2 \geq 2, \ldots, a_{s-1} - a_s \geq 2, \qquad a_s \geq s + 1,$$
$$a_{s+1} \geq a_{s+2} \geq \cdots \geq a_{2s} \geq 2,$$

then

$$a_1 + \cdots + a_{2s} \geq 2s^2 + 2s.$$

The interested reader can no doubt supply the details, or obtain them from the author.

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REFERENCES

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford (4th Edition).

[2] M. N. Hirschhorn, *Some partition theorems of the Rogers-Ramanujan type*, J. Combinatorial Analysis, **A27** (1979), 33–37.

[3] L. J. Slater, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc., **54** (1951–52), 147–167.

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