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A CLASS OF ARITHMETICAL EQUATIONS

BY

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1. This note arose essentially in attempting to generalize the following problem which appeared recently in the American Mathematical Monthly [3]: Determine the arithmetic function f satisfying

$$\sum_{d/n} f(d)f(n/d) = 1 \qquad (n = 1, 2, 3, \ldots).$$

The solution appears later in this note (in Section 3). Our interest here is to devise a method of finding all the solutions in f of the functional equation

$$f^{(r)} = g$$

for a given g, where $f^{(r)} = f \cdot f \cdot f \cdot \ldots \cdot f$ is the rth iterate of f with respect to a prescribed binary operation "$\cdot$" acting on the set S of all (complex valued) arithmetic functions.

Throughout sections 1–3 of this note we confine ourselves to the two important cases when the operation is the Dirichlet product

$$(\text{given by } (f \cdot g)(n) = \sum_{d/n} f(d)g(n/d))$$

or the unitary product

$$(\text{given by } (f \cdot g)(n) = \sum_{\substack{d/n \\ (d, n/d)=1}} f(d)g(n/d)).$$

We refer to $f^{(r)}$ as the r-th Dirichlet or unitary iterate of f as the case may be. Its values in the two cases are given respectively by

$$f^{(r)}(n) = \sum_{d_1 d_2 \ldots d_r = n} f(d_1)\, f(d_2) \ldots f(d_r), \qquad n = 1, 2, \ldots$$

and

$$f^{(r)}(n) = \sum_{\substack{d_1 d_2 \ldots d_r = n \\ (d_i, d_j)=1,\, i \neq j}} f(d_1)\, f(d_2) \ldots f(d_r).$$

2. We recall that an arithmetic function f is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)\,f(n)$ whenever $(m, n) = 1$. If $f^{(r)}$ is the Dirichlet or unitary iterate of f, it is known that the multiplicativity of f implies that of $f^{(r)}$. The direct converse of this result is false. However, we have the following conditional converse, which is useful for us in the sequel.

Theorem 1. *Let $f^{(r)}$ be the r-th Dirichlet or unitary iterate of f. Then the multiplicativity of $f^{(r)}$ implies the multiplicativity of f if and only if $f(1) = 1$.*

Proof. The "only if" part is trivial, and so we proceed to the proof of the "if" part. We consider only the Dirichlet case, the unitary case being similar.

Let $P(n)$ denote the property that for any positive integers a and b for which $(a, b) = 1$ and $ab = n$, we have $f(n) = f(a)\,f(b)$. Obviously $P(1)$ holds, and we now proceed by induction on n. Assume that $P(1), P(2), \ldots, P(n-1)$ all hold, n being ≥ 2, and consider any two natural numbers a and b such that $ab = n$. We exclude the trivial case when a or b is unity. We have

$$f^{(r)}(n) = f^{(r)}(ab) = \sum_{\substack{a_1 \ldots a_r = a \\ b_1 \ldots b_r = b}} f(a_1 b_1) \ldots f(a_r b_r), \tag{2.1}$$

and, by the induction hypothesis, $f(a_i b_i) = f(a_i)\,f(b_i)$ for all $a_i \mid a$ and $b_i \mid b$ except possibly for $a_i = a$, $b_i = b$. Thus (2.1) yields

$$\begin{aligned} f^{(r)}(n) = &\sum_{a_1 \ldots a_n = a} f(a_1) \ldots f(a_r) \sum_{b_1 \ldots b_r = b} f(b_1) \ldots f(b_r) + \\ &+ r(f(1))^{r-1} f(ab) - (f(1))^{2r-2} f(a)\,f(b) = \\ &= f^{(r)}(a)\,f^{(r)}(b) + r(f(ab) - f(a)\,f(b)) \end{aligned}$$

on using $f(1) = 1$. Since $f^{(r)}(n) = f^{(r)}(a)\,f^{(r)}(b)$, it follows that $f(ab) = f(a)\,f(b)$, thus establishing the truth of $P(n)$ and completing the proof.

Remark. If $f^{(r)}$ is replaced by $f_1 \cdot f_2 \cdot \ldots \cdot f_r$, in which at least two of the functions $f_1, f_2, \ldots, f_r$ are distinct, the theorem fails. Specifically, suppose $F = f_1 \cdot \ldots \cdot f_r$ $(r > 1)$ and not all the functions $f_1, \ldots, f_r$ are identical. Then the conditions (i) F is multiplicative; (ii) $f_1(1) = \ldots = f_r(1) = 1$, do not ensure the multiplicativity of any of the functions $f_1, \ldots, f_r$. In fact, given a multiplicative function F and integers r, s such that $r > 1$, $0 < s \leq r$, F can be expressed in an infinity of ways as the Dirichlet or unitary product of r functions $f_1, \ldots, f_r$, of which at least s are non-multiplicative;

and this result holds even if we prescribe that $f(1) = \ldots = f_r(1) = 1$. (This could be generalized in various ways). To see this, we first recall that the (Dirichlet or unitary) inverse f^{-1} of a function f, which is uniquely defined by $(f \cdot f^{-1})(n) = 1$ or 0 according as $n = 1$ or $n > 1$, exists if and only if $f(1) \neq 0$. Thus we can choose $f_1, \ldots, f_{r-1}$ to be *any* non-multiplicative functions subject only to the condition that $f_i(1) \neq 0$ $(i = 1, \ldots, r-1)$. Then we choose for f_r the function $f_1^{-1} \cdot \ldots \cdot f_{r-1}^{-1} \cdot F$. (Vide, for example, [1] and [2] in this connection).

However, we can show

Theorem 2. *Let $F = f_1 \cdot \ldots \cdot f_r$ $(r > 1$ be multiplicative and $f_1(1) = \ldots = f_r(1) = 1$. Also let $f_i(ab) - f_i(a)\, f_i(b) \geq 0$ whenever $(a, b) = 1$ for all i $(i = 1, 2, \ldots, r)$. (Alternately we can replace ≥ 0 by ≤ 0). Then F is multiplicative if and only if f_i is multiplicative $(i = 1, \ldots, r)$.*

The proof of this is along the same lines as that of theorem 1.

In the sequel, we denote the r-th roots of unity by $w_1 = 1, w_2, \ldots, w_r$, r being a positive integer. If a is a complex number, the function af is given by

$$(af)(n) = af(n) \qquad (n = 1, 2, \ldots).$$

Theorem 3. *Let g be a given multiplicative function. The equation $f^{(r)} = g$ has r solutions, of which only one is multiplicative. Denoting this solution by h, all the solutions are given by $f = w_1 h, w_2 h, \ldots, w_r h$.*

Proof. From $f^{(r)} = g$, one has $1 = g(1) = (f(1))^r$, so that $f(1)$ has the values $1 = w_1, w_2, \ldots, w_r$. Let the solution corresponding to the case $f(1) = 1$ be denoted by $f = h$. By theorem 1, h is a multiplicative function. Let f_i $(i = 1, \ldots, r)$ be the solution for which $f_i(1) = w_i$ (so that $f_1 = h$).

The theorem is proved if we show that for $i = 1, 2, \ldots, r$, we have

$$f_i(n) = w_i h(n) \tag{2.1}$$

for $n = 1, 2, \ldots$.

Obviously (2.1) holds for $n = 1$; we keep i fixed and assume that (2.1) holds for $n = 1, 2, \ldots, m-1)$ $(m > 2)$ and use mathematical induction. The equations which determine $h(m)$ and $f_i(m)$ are respectively (taking the case of Dirichlet Products):

$$mh(m)(h(1))^{m-1} + \sum_{\substack{d_1 \ldots d_r = m \\ 1 \leq d_k < m \\ (k=1, \ldots, r)}} h(d_1) \ldots h(d_r) = g(m) \tag{2.2}$$

and

$$mf_i(m)(f_i(1))^{m-1} + \sum_{\substack{d_1 \dots d_r = m \\ 1 \le d_k < m \\ (k=1,\dots,r)}} f_i(d_1) \dots f_i(d_r) = g(m). \qquad (2.3)$$

Noting that $f_i(d_k) = w_i h(d_k)$ $(1 \le d_k < m)$ by the induction hypothesis, and that $w_i^r = 1$, we observe that each term in the summation in the left member of (2.2) is in fact equal to the corresponding term of the summation in the left member of (2.3). Hence, equating the left members of (2.2) and (2.3), we obtain

$$mw_i^{r-1} f_i(m) = mh(m),$$

thus giving $f_i(m) = w_i h(m)$, and completing the induction proof.

The proof for the unitary case is similar. Actually in this case we can say a little more, namely.

Theorem 4. *If g is multiplicative and $f^{(r)}$ the r-th unitary iterate of r, the equation $f^{(r)} = g$ has r solutions in f given by*

$$g = w_k h \qquad (k = 1, 2, \dots),$$

where h is a multiplicative function determined by $h(p^k) = (1/r)g(p^k)$, p being an arbitrary prime and $k = 1, 2, \dots$.

Proof. This follows from Theorem 3 together with the fact that for $k > 0$,

$$g(p^k) = f^{(r)}(p^k) = r(f(1))^{r-1} f(p^k).$$

Similarly one can show

Theorem 5. *If g is any arithmetic function $\not\equiv 0$, the equation $f^{(r)} = g$ has exactly r distinct solutions which can be represented in the form*

$$f = w_i h \qquad (i = 1, 2, \dots, r),$$

h being any one of the solutions. One of these solutions is a multiplicative function if and only if g is multiplicative.

Remark. This result can also be proved from group-theoretic considerations, in view of the fact that the set of all arithmetic functions form a group G (with respect to Dirichlet or unitary multiplication) with identity e given by $e(n) = 1$ or 0 according as $n = 1$ or $n > 1$; and that the set of multiplicative arithmetic functions form a subgroup of G. (See for example [1] and [2]).

3. *The use of generating series.* Given an arithmetic function g, its generating series $g_{(p)}(x)$ to the base p, a prime, is defined by

the formal power series

$$g_{(p)}(x) = \sum_{s=0}^{\infty} g(p^s)x^s.$$

If $f^{(r)} = g$, we then have, formally, in the case of Dirichlet products,

$$(f(x))^r = g_{(p)}(x).$$

Hence the "formal r-th roots" of the series denoted by $g_{(p)}(x)$ give the r values for $f_{(p)}(x)$, and hence for $f(p^k)$ $(k = 0, 1, 2, \ldots)$.

If g is multiplicative, the unique solution $f = h$ of $f^{(r)} = g$, which is multiplicative, is formally given by the equation

$$(1 + h(p)x + h(p^2)x^2 + \ldots)^r = (1 + g(p)x + g(p^2)x^2 + \ldots)$$

or,

$$1 + h(p)x + h(p^2)x^2 + \ldots = (1 + g(p)x + g(p^2)x^2 + \ldots)^{1/r}.$$

This relation is specially helpful in determining h when the series $1 + g(p)x + g(p^2)x^2 + \ldots$ adds up to a simple function, as illustrated by the following examples. Once h is determined, all the solutions of $f^{(r)} = g$ are, by Theorem 3, given by $w_i h$ $(i = 1, 2, \ldots, r)$.

Example 1. To determine f satisfying [3]

$$\sum_{d/n} f(d)f(n/d) = 1, \qquad n = 1, 2, \ldots .$$

Here,

$$g_{(p)}(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x},$$

so that

$$h_{(p)}(x) = \frac{1}{(1-x)^{\frac{1}{2}}}.$$

This gives for $k > 0$,

$$h(p^k) = \frac{1\cdot 3\cdot 5 \ldots (2k-1)}{2\cdot 4\cdot 6 \ldots (2k)} \tag{3.1}$$

for all primes p. The two solutions are $f = h$ and $f = -h$, and are completely determined by (3.1).

Example 2. To solve for f satisfying

$$n \sum_{d/n} f(d)f(n/d) = \sigma_2(n),$$

where $\sigma_2(n)$ denotes the sum of the squares of the (positive) divisors of n.

Here, $g(p^k) = \sigma_2(p^k)/p^k$ ($k = 0, 1, \ldots$).
A simple computation shows that

$$g_{(p)}(x) = \frac{1}{1 - \left(p + \frac{1}{p}\right) x + x^2} .$$

The two solutions are given by $f = h, -h$, where $h(1) = 1$ and, for $k > 0$, $h(p^k) = P_k(p + 1/p)$, where $P_n(x)$ is the Legendre polynomial of degree n.

Example 3. Let $P_n(x)$, $T_n(x)$ ($n = 1, 2, \ldots$) be the usual Legendre and Tchebichef polynomials with the convention $P_0(x) = 1$, $T_0(x) = \frac{1}{2}$; and $P_n(x) = T_n(x) = 0$ for $n < 0$. Let $g(n)$ be the multiplicative function given by

$$g(p^k) = 2(T_k(x) - T_{n-2}(x)),$$

where p is any prime and x is fixed (the same for all p). Then the solutions of the functional equation

$$\sum_{d/n} f(d)g(n/d) = g(n)$$

are given by $f(n) = h(n)$, $-h(n)$, where $h(n)$ is the multiplicative function determined by $h(p^k) = P_k(x) - P_{k-2}(x)$, $k = 0, 1, 2, \ldots$, p being an arbitrary prime.

Example 4. The equation

$$\sum_{\substack{d/n \\ (d,n/d)=1}} f(d)f(n/d) = 1, \qquad n = 1, 2, \ldots,$$

has the two solutions $f(n) = h(n)$, $-h(n)$, where

$$h(n) = \begin{cases} 1 & n = 1 \\ 2^{-w(n)} & n > 1, \end{cases}$$

where $w(n)$ denotes the number of distinct prime divisors of n.

4. *A further generalization.* Theorem 1, 2 and 3 can be generalized in various ways. For instance, we can take for $f^{(r)}$ the r-th iterate of f with respect to the operation of "K-product" [4, 5]. If K is any arithmetic function such that $K(1) = 1$ and for all positive integers a, b and c,

$$K((a, b))\, K((ab, c)) = K((a, bc))\, K((b, c)), \tag{4.1}$$

the K-product of two arithmetic functions f and g is defined as

$$\sum_{d_1 d_2 = n} f(d_1)\, g(d_2)\, K((d_1, d_2)).$$

The function K with the above restrictions should necessarily be multiplicative; and (4.1) ensures that the K-product is an associative operation. The proof that theorems 1, 2 and 3 are valid for K-products is left to the interested reader. It is easy to see that Dirichlet and unitary products are special cases of K-products.

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