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A CHARACTERIZATION OF INNER PRODUCT SPACES

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1. Several writers have recently been interested in the problem of finding necessary and sufficient conditions on the norm of a normed vector space E admitting multiplication by scalars (we confine ourselves to real scalars) in order that an inner product can be introduced in E. The earliest result is due to Jordon and Von Neumann [5] and can be stated thus:

A necessary and sufficient condition for introducing an inner product in E *is that for* $x, y \varepsilon$ E,

(a) $||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$

This has been improved upon by M. M. Day [1] who showed that it will suffice if

(b) $||x|| = ||y|| = 1$ implies $||x+y||^2 + ||x-y||^2 = 4$

Day also proved that another necessary and sufficient condition is "Pythagorous Theorem":

(c) $||x+y|| = ||x-y||$ implies

$$||x+y||^2 = ||x||^2 + ||y||^2$$

On the otherhand Ficken [2] obtained the condition

(d) $||x|| = ||y||$ implies $||x+ay|| = ||ax+y||$ for all *real numbers* a, another form of which is

(d^1) $||x+y|| = ||x-y||$ implies $||x+ay|| = ||x-ay||$ *for all real numbers* a

This has been improved upon by Lorch [6] who showed that it will suffice if (d^1) is satisfied for just one value of a, $a \neq 0, \pm 1$. Lorch has also given the following new conditions:

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(e) $||x|| = ||y||$ *implies* $||ax + a^{-1}y|| \geqslant ||x + y||$ *for all numbers a*

(e¹) $||ax + a^{-1}y|| \geqslant ||x + y||$ *for all real numbers a implies* $||x|| = |y||$.

The interest of these results, as Lorch himself says, lies in the fact that they involve inequalities in the norms rather than equalities. In this paper I will give two new and equivalent characterizations involving inequalities in the norm for the existence of an inner product in a normed linear space E.

2. *Theorem*(1) *Either of the two following equivalent conditions is necessary and sufficient for introducing an inner product in* E:

(f) $x, y \in E$, $||x + y|| = ||x - y||$ *implies* $||x + ay|| \geqslant ||x||$ *for all real values of a.*

(g) $x, y \in E$, $||x + ay|| \geqslant ||x||$ *for all real values of a implies* $||x + y|| = ||x - y||$

Proof. (i) (*f*) *implies* (*g*). Writing x and y in the place of $x + y$ and $x - y$, (f) and (g) take the form.

(f¹) $||x|| = ||y||$ *implies* $||x + \frac{1-a}{1+a} y|| \geqslant \frac{1}{|1+a|} \cdot ||x + y||$ *for all real values of a,*

(g¹) $||x + \frac{1-a}{1+a} y|| \geqslant \frac{1}{|1+a|} ||x + y||$ *for all real values of a implies* $||x|| = ||y||$.

Assume (f¹) to be true and let

$$(1.)\quad ||x + \frac{1-a}{1+a} y|| \geqslant \frac{1}{|1+a|} \cdot ||x + y|| \text{ for all } a.$$

(1) The theorem shows incidentally that the two definitions for orthogonality between two elements x, y of a normed linear space E considered by James [3], [4], viz., $||x + y|| = ||x - y||$, and $||x + ay|| \geqslant ||x||$ for all real numbers a, are equivalent if and only if an inner product exists in E.

To show $||x|| = ||y||$.

In (1), $x \neq o$, $y \neq o$, for otherwise (1) will not be satisfied for all a.

Put $y^1 = \dfrac{||x||}{||y||} y$; then $||y^1|| = ||x||$. Thus (f^1)

$$\text{gives } ||x + \frac{1-a}{1+a} y^1|| \geqslant \frac{1}{1-a} ||x + y^1||$$

$$\text{(i.e.) } ||x + \frac{1-a}{1+a} \cdot \frac{||x||}{||y||} y|| \geqslant \frac{1}{|1+a|} \cdot ||x + \frac{||x||}{||y||} y||$$

$$\geqslant \frac{1}{|+a|} \cdot \frac{||x|| + ||y||}{2||y||} \cdot ||x+y||, \text{ by (1)}$$

This being true for all values of a, choose a such that

$$\frac{1-a}{1+a} \cdot \frac{||x||}{||y||} = 1, \text{ so that } 1+a = \frac{2||x||}{||x|| + ||y||}$$

The above inequality becomes

$$||x+y|| \geqslant \frac{||x|| + ||y||}{2||x||} \cdot \frac{||x|| + ||y||}{2||y||} ||x+y||$$

Hence $(||x|| + ||y||)^2 \leqslant 4||x|| \cdot ||y||$,
or $||x|| = ||y||$.

(ii) (g) *implies* (f). We first note the following

Lemma. If $\Phi(a)$ is a convex function of the real variable a in a given interval and if $\Phi(a) = \Phi(b)$, then

(1) $\Phi(c) \leqslant \Phi(a)$ if c lies in the interval (a, b).

(2) $\Phi(c) \geqslant \Phi(a)$ otherwise.

If c lies in (a, b), $c = \lambda a + (1-\lambda) b$, $0 \leqslant \lambda \leqslant 1$, so that
$\Phi(c) \leqslant \lambda \; \Phi(a) + (1-\lambda) \; \varphi(b) = \Phi(a)$

If c is outside (a, b), say $a < b < c$, then $b = \lambda a + (1-\lambda) c$, $0 \leqslant \lambda \leqslant 1$, $\Phi(b) \leqslant \lambda \; \Phi(a) + (1-\lambda) \; \Phi(c)$, or $\Phi(c) \geqslant \Phi(a)$

Now to prove that (g) implies (f), let us assume the truth of (g) and let

(2) $$||x+y|| = ||x-y||$$

To show $||x+ay|| \geqslant ||x||$ for all a.

Put $\Phi(a) = ||x+ay||$; then Φ (a) is continuous in a and $\to \infty$ with $|a|$. Also by using the triangle inequality of the norm, $\Phi(a)$ is a convex function.

Now $\Phi(a)$ must have a finite lower bound which it must attain, say at $a = a_o$, atleast once. Our result is proved if we show that it is attained at $a = 0$.

Now we can assume $|a_o| < 1$; for if $|a_o| \geqslant 1$, by the lemma, we see that since $\varphi(1) = \varphi(-1)$ by (2), we will have $\varphi(1) = \varphi(a_o) = \varphi(-1)$; and so also $\varphi(o) = \varphi(a_o)$, proving the required result. So we assume $|a_o| < 1$. Now

$$||x+ay|| \geqslant ||x+a_oy||$$

or $||(x+a_oy) + (a-a_o)y|| \geqslant ||x+a_oy||$,

$$-\infty \leqslant a - a_o < \infty$$

By applying property (g), we have

$$||x+a_oy+y|| = ||x+a_oy-y||,$$

(3) *or* $\varphi(a_o+1) = \varphi(a_o-1)$.

For definiteness let us assume as $> o$ (if $a_o < 0$, the argument is similar), then

$$-1 < a_o - 1 < o < a_o < 1 < a_o + 1.$$

Using (3) and the lemma, we have since $a_o - 1 < 1 < a_o + 1$,

(4) $$\varphi(a_o-1) \geqslant \varphi(1)$$

Again since $-1 < a_o - 1 < 1$, and $\varphi(-1) = \varphi(1)$, we have

$$\varphi(a_o-1) \leqslant \varphi(1)$$

Hence $\varphi(-1) = \varphi(1) = \varphi(a_o-1) = \varphi(a_o+1)$

Next, since $-1 < a_o < a_o + 1$, we have

$$\varphi(a_o) \leqslant \varphi(-1) = \varphi(1)$$

and since $a_o < 1 < a_o + 1$,

$$\varphi(a_o) \geqslant \varphi(1)$$

Thus $\varphi(a_o) = \varphi(1) = \varphi(-1)$.

By the lemma, we have also $\varphi(o) = \varphi(a_o) = \varphi(1)$: and hence the result. We will finally show that conditions (f) and (g) are necessary and sufficient for the existence of inner product. That the conditions are necessary is easily proved. We will show that (f), (and hence (g)) is a sufficient condition for introducing an inner product in E.

Putting $a = \dfrac{k^2-1}{k^2+1}$ condition (f) gives

$$||x+y|| = ||x-y|| \text{ implies } ||x + \frac{k^2-1}{k^2+1} y|| \geqslant ||x||$$

$$\geqslant \frac{2|k|}{k^2+1} ||x||, \text{ since } \frac{2|k|}{k^2+1} < 1$$

(i.e.) $||k x^1 + k^{-1} y^1|| \geqslant ||x^1 + y^1||$ for all real numbers k, where $x^1 = x + y$, $y^1 = x - y$. Hence (f) implies (c), and hence the result.

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