I. Introduction. For any arithmetic function \( f(n) \), we denote its iterates as follows:

\[
    f_1(n) = f(n); \quad f_k(n) = f_1[f_{k-1}(n)] \quad (k > 1).
\]

Let \( \sigma(n) \) and \( \sigma^*(n) \) denote, respectively, the sum of the divisors of \( n \), and the sum of its unitary divisors, where we recall that \( d \) is called a unitary divisor of \( n \) if \( (d,n/d) = 1 \). Makowski and Schinzel [3] proved that

\[
    \liminf \frac{\sigma_2(n)}{n} = 1,
\]

and conjectured that

\[
    \liminf \frac{\sigma_k(n)}{n} < \infty \quad \text{for every } k.
\]

This is not proved even for \( k = 3 \). On the other hand, Erdős [2] stated that if we neglect a sequence of density zero, then

\[
    \frac{\sigma_k(n)}{\sigma_{k-1}(n)} = (1 + o(1)) \, e^{\sqrt[3]{\log \log \log n}}.
\]

This implies, in particular, that

\[
    \frac{\sigma_2(n)}{\sigma_1(n)} \to \infty
\]

on a set of density unity.

In contrast to this, we show here the following result.

Theorem 1.

\[
    \frac{\sigma^*_2(n)}{\sigma^*_1(n)} \to 1 \quad \text{on a set of density unity}.
\]
2. Some lemmas. The proof makes use of the following lemmas. Throughout what follows, \( h, q, r, r_1, r_2 \) represent primes, and \( \epsilon, \eta \) small positive numbers. Almost all \( n < x \) will mean: all but \( o(x) \) integers \( n \leq x \).

Lemma 1. For almost all \( n < x \), every \( p < (\log \log x)^{1-\epsilon} \) satisfies \( p^2 | \sigma^*(n) \).

Lemma 2. For almost all \( n < x \) and for any given \( \eta \), we have

\[
\sum_{\substack{p \mid \sigma^*(n) \\ p > (\log \log x)^{1+\epsilon}}} \frac{1}{p} < \eta,
\]

where \( \epsilon = \epsilon(\eta) > 0 \) is sufficiently small.

Lemma 3. For almost all \( n < x \) and all \( p < t \) (\( t \) fixed but arbitrary),

\( p^\alpha | \sigma^*(n) \)

for every fixed \( \alpha \).

We only outline the proofs of the lemmas and the theorem.

Proof of Lemma 1. For a given \( p < (\log \log x)^{1-\epsilon} \) for which \( p | \sigma^*(n), n < x \), it is enough if we show that there are at least two primes \( r_1, r_2 \) such that

\( r_1 \equiv r_2 \equiv -1 \pmod{p} \),

and

\( r_1 | n, \quad r_2 | n, \quad r_1^2 | n, \quad r_2^2 | n. \)

For this purpose we use the Page-Walfisz-Siegel formula for primes in arithmetic progression (Pracher [6], p. 320) which states that if \( \pi(a,d,y) \) denotes the number of primes \( \equiv a \pmod{d} \) and \( \leq y \), then for \( (a,d) = 1 \),

\[
\pi(a,d,y) = (1 + o(1)) \frac{y}{\phi(d) \log y}
\]

uniformly in \( a \) and \( d \) for \( d < (\log y)^t \) for every fixed \( t \). Hence, for primes \( r \) such that \( r | n, \ r \equiv -1 \pmod{p} \), we have
\[ \sum_{r = -1 \pmod{p}} \frac{1}{r} > c (\log \log x)^\epsilon. \]

\[ \log \log x < r < x \]

Hence we easily obtain by the sieve of Brun or Selberg that the number of integers \( n < x \) which are divisible by just one prime is less than \( x \exp(-c (\log \log x)^\epsilon) \). There are fewer than \( (\log \log x)^{1-\epsilon} \), and \( (\log \log x)^{1-\epsilon} \exp(-c (\log \log x)^\epsilon) = o(x) \), and the number of integers which are divisible by the square of a prime \( > \log \log x \) is \( o\left(\frac{x}{\log \log x}\right) \). Thus these numbers can be ignored. Thus Lemma 1 is proved.

**Proof of Lemma 2.** We consider the sum

\[ S = \sum_{n=1}^{x} \sum_{p \mid \sigma^*(n) \mid p > (\log \log x)^{1+\epsilon}} \frac{1}{p}. \]

For a fixed \( p \), we see that every prime \( r \) such that \( r = -1 \pmod{p} \), \( r \mid n \), contributes a factor \( p \) to \( \sigma^*(n) \). Since the number of integers \( n < x \) for which \( r \mid n \) is \( \left\lfloor \frac{x}{r} \right\rfloor \), it follows that for a given \( p \) the number of times the term \( \frac{1}{p} \) occurs in the sum \( S \) corresponding to each prime \( r = -1 \pmod{p} \) is less than \( \left\lfloor \frac{x}{r} \right\rfloor \). Also, on using the Brun-Titchmarsh estimate for primes in arithmetic progression [6, p. 320] we have

\[ \sum_{r = -1 \pmod{p}} \left\lfloor \frac{x}{r} \right\rfloor < \frac{c x \log \log x}{p}. \]

Hence

\[ S < c x \log \log x \sum_{p > (\log \log x)^{1+\epsilon}} \frac{1}{p^2} = o(x). \]

**Proof of Lemma 3.** Given a \( p < t \), we see, on using the sieve of Eratosthenes and the fact that

\[ \sum_{r = -1 \pmod{p}} \frac{1}{r} = \infty, \]

\( r = -1 \pmod{p} \)
that the number of integers \( n \leq x \) such that \( n \) is divisible by at most \( j \) primes \( q \) of the form \( q \equiv -1 \pmod{p} \), each of them occurring to the first power in \( n \), is \( o(x) \), \( j \) being an arbitrary positive integer. Hence the number of such integers \( n \leq x \) is \( o(x) \). Since for each such \( n \) we have \( p^j | \sigma^*(n) \), the lemma follows at once.

3. Proof of the theorem. Let \( \eta \) be chosen arbitrarily small and then keep it fixed. We shall then choose \( t \) and \( \alpha = \alpha(t) \) sufficiently large so that

\[
\prod_{p < t} \left(1 + \frac{1}{p^{\alpha}}\right) < 1 + \eta
\]

and

\[
\prod_{p \geq t} \left(1 + \frac{1}{p^{2}}\right) < 1 + \eta.
\]

The latter inequality is possible because of the convergence of \( \prod(1 + \frac{1}{p^{2}}) \).

Since almost all \( n < x \) satisfy Lemmas 1, 2, 3, we have for almost all \( n \),

\[
\frac{\sigma^*(n)}{\sigma_1(n)} \leq \prod_{p \leq t} \left(1 + \frac{1}{p^{\alpha}}\right) \prod_{p > t} \left(1 + \frac{1}{p^{2}}\right) \cdot \prod_{(\log \log x)^{1-\epsilon} < p < (\log \log x)^{1+\epsilon}} \left(1 + \frac{1}{p}\right),
\]

on noting that

\[
(\log \log x)^{1-\epsilon} \leq \sum_{\frac{1}{p} < (\log \log x)^{1+\epsilon}} \frac{1}{p} < \eta
\]

for a suitably chosen \( \epsilon = \epsilon(\eta) \).

Combining Lemma 2 and the result (3.4), we get

\[
\prod_{p \sigma^*(n) \leq t} \left(1 + \frac{1}{p}\right) < 1 + \eta.
\]
It then follows from (3.3) that for almost all \( n \), i.e., except for values of \( n \) with density zero,
\[
\frac{\sigma_2^*(n)}{\sigma_1^*(n)} < 1 + \eta,
\]
and the proof of the theorem is complete. Our theorem implies that \( \frac{\sigma_2^*(n)}{n} \) has the same distribution function as \( \frac{\sigma_1^*(n)}{n} \).

4. Some remarks and problems. Let \( \phi^*(n) \) be the unitary analogue of Euler's totient function (see E. Cohen [1]). Then \( \phi^*(n) \) has the evaluation
\[
\phi^*(n) = \prod_{\mathfrak{p} \mid \mid n} (\mathfrak{p}^a - 1).
\]

Following the method of proof of Theorem 1, we can show that
\[
\frac{\phi_2^*(n)}{\phi_1^*(n)} \to 1 \quad (\phi_1^*(n) = \phi^*(n))
\]
except for a sequence of values of \( n \) of density zero. We shall not give the details of proof.

Let \( R = R(n) \) be the smallest integer such that \( \phi_R(n) = 1 \). This function was first considered by S. S. Pillai [5] who proved that
\[
\frac{\log(n/2)}{\log 3} + 1 \leq R(n) \leq \frac{\log n}{\log 2} + 1.
\]

Others who considered this function include Niven [4], Shapiro [7] and Subbarao [8].

Let
\[
T(n) = \phi_1(n) + \phi_2(n) + \cdots + \phi_R(n).
\]
Since \( \phi_2(n) = o(\phi_1(n)) \) for almost all \( n \), and \( \phi_j(n) \) is even for \( j \geq 1 \), we easily obtain that for almost all \( n \)
\[
T(n) = (1 + o(1))\phi(n),
\]
so that \( T(n) < n \) for almost all \( n \).
There are many problems left about $T(n)$ and we state a few of them below.

Denote by $F(x, c)$ the number of integers $n \leq x$ for which $T(n) > cn$. For every $1 < c < 3/2$ we have for every $t > 0$ and $\epsilon > 0$, if $x > x_0 = x_0(c, t, \epsilon)$,

\begin{equation}
\frac{x}{\log x} \left( \log \log x \right)^t < F(x, 1+c) < \frac{x}{(\log x)^{1-\epsilon}}.
\end{equation}

This follows easily from Theorem 1 of [2]. Further we have

\begin{equation}
F(x, 1) = (c + o(1)) \frac{x}{\log \log \log \log x}.
\end{equation}

The proof of (4.2) can be obtained by the methods used in this paper and by those of [2].

It seems likely that for $1 < c_1 < c_2 < \frac{3}{2}$,

\[ \lim_{x \to \infty} \frac{F(x, 1+c_1)}{F(x, 1+c_2)} = \infty. \]

Put

\[ L = \lim_{n} \frac{T(n)}{n}. \]

Trivially $L \leq 2$ ($L = 2$ if there are infinitely many Fermat primes). It is easy to show that

\[ \lim_{n} \frac{T(2n)}{2n} = 1. \]

We can show that $T(n) > \frac{3n}{2}$ for infinitely many $n$, which implies $L \geq \frac{3}{2}$. We cannot show that $L > \frac{3}{2}$.

Equation (3) of Theorem 1 of [2] implies that for $c > \frac{3}{2}$ and every $\epsilon > 0$,

\[ F(x, c) = o\left( \frac{x}{(\log x)^{2-\epsilon}} \right). \]

Probably,

\[ F(x, \frac{3}{2}) = o\left( \frac{x}{\log x} \right). \]
but we have not worked out the details.

Some other questions that are still unanswered are the following:

(i) Does \( \frac{R(n)}{\log n} \) have a distribution function?

(ii) Does \( \frac{R(n)}{\log n} \) approach a limit for almost all \( n \)? If this limit exists, is it equal to \( \frac{1}{\log 2} \) or \( \frac{1}{\log 3} \)?

Similar questions arise in the case of the function \( R^* = R^*(n) \) defined as the smallest integer such that \( \varphi^*_R(n) = 1 \). Here \( \varphi^*_R(n) \) is the unitary analogue of the Euler totient, introduced by Eckford Cohen [1], which is defined as the multiplicative function for which \( \varphi^*_R(p^k) = p^k - 1 \) for all primes \( p \) and all positive integers \( k \). We do not even know of any nontrivial estimate for \( R^*(n) \). Probably \( R^*(n) = o(n^\epsilon) \) for every \( \epsilon > 0 \). It is not clear to us at present if \( R^*(n) < c \log n \) has infinitely many solutions for some \( c > 0 \).

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