ON THE ITERATES OF SOME ARITHMETIC FUNCTIONS

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1. Introduction. For any arithmetic function $f(n)$, we denote its iterates as follows:

$$f_1(n) = f(n); \quad f_k(n) = f_1[f_{k-1}(n)] \quad (k > 1).$$

Let $\sigma(n)$ and $\sigma^*(n)$ denote, respectively, the sum of the divisors of n, and the sum of its unitary divisors, where we recall that d is called a unitary divisor of n if $(d,n/d) = 1$. Makowski and Schinzel [3] proved that

$$\liminf \frac{\sigma_2(n)}{n} = 1,$$

and conjectured that

$$\liminf \frac{\sigma_k(n)}{n} < \infty \quad \text{for every } k.$$

This is not proved even for $k = 3$. On the other hand, Erdős [2] stated that if we neglect a sequence of density zero, then

$$\frac{\sigma_k(n)}{\sigma_{k-1}(n)} = (1 + o(1))k\, e^{\gamma} \log\log\log n.$$

This implies, in particular, that

$$\frac{\sigma_2(n)}{\sigma_1(n)} \to \infty$$

on a set of density unity.

In contrast to this, we show here the following result.

Theorem 1.

$$\frac{\sigma_2^*(n)}{\sigma_1^*(n)} \to 1 \quad \text{on a set of density unity.}$$

2. Some lemmas. The proof makes use of the following lemmas. Throughout what follows, h, q, r, r_1, r_2 represent primes, and ϵ, η small positive numbers. Almost all $n < x$ will mean: all but $o(x)$ integers $n \le x$.

Lemma 1. For almost all $n < x$, every $p < (\log\log x)^{1-\epsilon}$ satisfies $p^2 | \sigma^*(n)$.

Lemma 2. For almost all $n < x$ and for any given η, we have

$$\sum_{\substack{p | \sigma^*(n) \\ p > (\log\log x)^{1+\epsilon}}} \frac{1}{p} < \eta,$$

where $\epsilon = \epsilon(\eta) > 0$ is sufficiently small.

Lemma 3. For almost all $n < x$ and all $p < t$ (t fixed but arbitrary),

$$p^{\alpha} | \sigma^*(n)$$

for every fixed α.

We only outline the proofs of the lemmas and the theorem.

Proof of Lemma 1. For a given $p < (\log\log x)^{1-\epsilon}$ for which $p | \sigma_2^*(n)$, $n < x$, it is enough if we show that there are at least two primes r_1, r_2 such that

$$r_1 \equiv r_2 \equiv -1 \pmod p,$$

and

$$r_1 | n, \quad r_1^2 | n, \quad r_2 | n, \quad r_2^2 | n.$$

For this purpose we use the Page-Walfisz-Siegel formula for primes in arithmetic progression (Pracher [6], p. 320) which states that if $\pi(a,d,y)$ denotes the number of primes $\equiv a \pmod d$ and $\le y$, then for $(a,d) = 1$,

$$\pi(a,d,y) = (1 + o(1))\frac{y}{\varphi(d)\log y}$$

uniformly in a and d for $d < (\log y)^t$ for every fixed t. Hence, for primes r such that $r | n$, $r \equiv -1 \pmod p$, we have

$$\sum_{\substack{r \equiv -1 \pmod p \\ \log\log x < r < x}} \frac{1}{r} > c(\log\log x)^{\epsilon}.$$

Hence we easily obtain by the sieve of Brun or Selberg that the number of integers $n < x$ which are divisible by just one prime is less than $x\exp(-c(\log\log x)^{\epsilon})$. There are fewer than $(\log\log x)^{1-\epsilon}$ primes $< (\log\log x)^{1-\epsilon}$, and $(\log\log x)^{1-\epsilon} x\exp(-c(\log\log x)^{\epsilon}) = o(x)$, and the number of integers which are divisible by the square of a prime $> \log\log x$ is $o\left(\frac{x}{\log\log x}\right)$. Thus these numbers can be ignored. Thus Lemma 1 is proved.

<u>Proof of Lemma</u> 2. We consider the sum

$$S = \sum_{n=1}^{x} \sum_{\substack{p \mid \sigma^*(n) \\ p > (\log\log x)^{1+\epsilon}}} \frac{1}{p}.$$

For a fixed p, we see that every prime r such that $r \equiv -1 \pmod p$, $r \mid n$, contributes a factor p to $\sigma^*(n)$. Since the number of integers $n < x$ for which $r \mid n$ is $\left[\frac{x}{r}\right]$, it follows that for a given p the number of times the term $\frac{1}{p}$ occurs in the sum S corresponding to each prime $r \equiv -1 \pmod p$ is less than $\left[\frac{x}{r}\right]$. Also, on using the Brun-Titchmarsh estimate for primes in arithmetic progression [6, p. 320] we have

$$\sum_{r \equiv -1 \pmod p} \left[\frac{x}{r}\right] < \frac{c\,x\log\log x}{p}.$$

Hence

$$S < c\,x\log\log x \sum_{p > (\log\log x)^{1+\epsilon}} \frac{1}{p^2} = o(x).$$

<u>Proof of Lemma</u> 3. Given a $p < t$, we see, on using the sieve of Eratosthenes and the fact that

$$\sum_{r \equiv -1 \pmod p} \frac{1}{r} = \infty,$$

that the number of integers $n \le x$ such that n is divisible by at most j primes q of the form $q \equiv -1 \pmod p$, each of them occurring to the first power in n, is $o(x)$, j being an arbitrary positive integer. Hence the number of such integers $n \le x$ is $o(x)$. Since for each such n we have $p^j \mid \sigma^*(n)$, the lemma follows at once.

3. Proof of the theorem. Let η be chosen arbitrarily small and then keep it fixed. We shall then choose t and $\alpha = \alpha(t)$ sufficiently large so that

$$\prod_{p < t}{}' \left(1 + \frac{1}{p^\alpha}\right) < 1 + \eta \tag{3.1}$$

and

$$\prod_{p \ge t}{}' \left(1 + \frac{1}{p^2}\right) < 1 + \eta. \tag{3.2}$$

The latter inequality is possible because of the convergence of $\prod\left(1 + \frac{1}{p^2}\right)$.

Since almost all $n < x$ satisfy Lemmas 1, 2, 3, we have for almost all n,

$$\frac{\sigma_2^*(n)}{\sigma_1^*(n)} \le \prod_{p \le t}{}' \left(1 + \frac{1}{p^\alpha}\right) \prod_{p > t}{}' \left(1 + \frac{1}{p^2}\right) \cdot \prod_{(\log\log x)^{1-\epsilon} < p < (\log\log x)^{1+\epsilon}}{}' \left(1 + \frac{1}{p}\right), \tag{3.3}$$

on noting that

$$\sum_{(\log\log x)^{1-\epsilon} < p < (\log\log x)^{1+\epsilon}} \frac{1}{p} < \eta \tag{3.4}$$

for a suitably chosen $\epsilon = \epsilon(\eta)$.

Combining Lemma 2 and the result (3.4), we get

$$\prod_{\substack{p > t \\ p \mid \sigma^*(n) \\ p^2 \nmid \sigma^*(n)}}{}' \left(1 + \frac{1}{p}\right) < 1 + \eta.$$

It then follows from (3.3) that for almost all n, i.e., except for values of n with density zero,

$$\frac{\sigma_2^*(n)}{\sigma_1^*(n)} < 1 + \eta,$$

and the proof of the theorem is complete. Our theorem implies that $\sigma_2^*(n)/n$ has the same distribution function as $\sigma_1^*(n)/n$.

4. <u>Some remarks and problems</u>. Let $\varphi^*(n)$ be the unitary analogue of Euler's totient function (see E. Cohen [1]). Then $\varphi^*(n)$ has the evaluation

$$\varphi^*(n) = \prod_{p^a \| n} (p^a - 1).$$

Following the method of proof of Theorem 1, we can show that

$$\frac{\varphi_2^*(n)}{\varphi_1^*(n)} \to 1 \qquad (\varphi_1^*(n) = \varphi^*(n))$$

except for a sequence of values of n of density zero. We shall not give the details of proof.

Let $R = R(n)$ be the smallest integer such that $\varphi_R(n) = 1$. This function was first considered by S. S. Pillai [5] who proved that

$$\frac{\log(n/2)}{\log 3} + 1 \le R(n) \le \frac{\log n}{\log 2} + 1.$$

Others who considered this function include Niven [4], Shapiro [7] and Subbarao [8].

Let

$$T(n) = \varphi_1(n) + \varphi_2(n) + \cdots + \varphi_R(n).$$

Since $\varphi_2(n) = o\left(\varphi_1(n)\right)$ for almost all n, and $\varphi_j(n)$ is even for $j \ge 1$, we easily obtain that for almost all n

$$T(n) = (1 + o(1))\varphi(n),$$

so that $T(n) < n$ for almost all n.

There are many problems left about $T(n)$ and we state a few of them below.

Denote by $F(x,c)$ the number of integers $n \le x$ for which $T(n) > cn$. For every $1 < c < 3/2$ we have for every $t > 0$ and $\epsilon > 0$, if $x > x_o = x_o(c,t,\epsilon)$,

$$\frac{x}{\log x}(\log\log x)^t < F(x,1+c) < \frac{x}{(\log x)^{1-\epsilon}}. \tag{4.1}$$

This follows easily from Theorem 1 of [2]. Further we have

$$F(x,1) = \Big(c + o(1)\Big)\frac{x}{\log\log\log\log x}. \tag{4.2}$$

The proof of (4.2) can be obtained by the methods used in this paper and by those of [2].

It seems likely that for $1 < c_1 < c_2 < \frac{3}{2}$,

$$\lim_{x\to\infty} F(x,1+c_1)/F(x,1+c_2) = \infty.$$

Put

$$L = \overline{\lim}\, \frac{T(n)}{n}.$$

Trivially $L \le 2$ ($L = 2$ if there are infinitely many Fermat primes). It is easy to show that

$$\overline{\lim}\, \frac{T(2n)}{2n} = 1.$$

We can show that $T(n) > \frac{3n}{2}$ for infinitely many n, which implies $L \ge \frac{3}{2}$. We cannot show that $L > \frac{3}{2}$.

Equation (3) of Theorem 1 of [2] implies that for $c > \frac{3}{2}$ and every $\epsilon > 0$,

$$F(x,c) = o\Big(\frac{x}{(\log x)^{2-\epsilon}}\Big).$$

Probably,

$$F(x,\tfrac{3}{2}) = o\Big(\frac{x}{\log x}\Big),$$

but we have not worked out the details.

Some other questions that are still unanswered are the following:

(i) Does $\frac{R(n)}{\log n}$ have a distribution function?

(ii) Does $\frac{R(n)}{\log n}$ approach a limit for almost all n? If this limit exists, is it equal to $\frac{1}{\log 2}$ or $\frac{1}{\log 3}$?

Similar questions arise in the case of the function $R^* = R^*(n)$ defined as the smallest integer such that $\varphi_{R^*}(n) = 1$. Here $\varphi^*(n)$ is the unitary analogue of the Euler totient, introduced by Eckford Cohen [1], which is defined as the multiplicative function for which $\varphi^*(p^k) = p^k - 1$ for all primes p and all positive integers k. We do not even know of any nontrivial estimate for $R^*(n)$. Probably $R^*(n) = o(n^\epsilon)$ for every $\epsilon > 0$. It is not clear to us at present if $R^*(n) < c \log n$ has infinitely many solutions for some $c > 0$.

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