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ON THE REPRESENTATION OF FRACTIONS AS SUM AND DIFFERENCE OF THREE SIMPLE FRACTIONS

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1. Introduction.

The conjecture that every fraction $\frac{4}{n}$ $(n > 1)$ can be expressed as the sum of three simple fractions was originally stated by P. Erdős and E.G. Straus in 1948. Since then, a considerable amount of attention has been given to this problem and a great deal of numerical evidence has been compiled in support of the same. For instance, Yamamoto [8] verified the conjecture for all $n < 10^7$. It is also known that the conjecture holds for almost all n, and in fact, the number of positive integers $n \leq x$ for which the conjecture fails is

$$0(x \exp[-c(\log x)^{2/3}])$$

c being a positive absolute constant >3. (See, for example, Mordell [1], Ch. 10, for an account of the problem).

A. Schinzel and W. Sierpinski [4] posed the general question whether for every positive integer a, there exists an $N(a)$ so that

$$\frac{a}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

has positive integral solutions for every $n > N(a)$. R.C. Vaughan [7] proved in 1970 that for any given $a > 0$, the number of $n \leq x$ for which a/n does not have the desired representation is

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$$O(x \exp[-c(a)(\log x)^{2/3}]) ,$$

$c(a) > 3$ being a positive number depending on a.

A more tractable problem, also posed by A. Schinzel in 1956, is whether

$$\frac{a}{n} = \frac{1}{x} \pm \frac{1}{y} \pm \frac{1}{z} \tag{1.2}$$

has integral solutions x, y, z for a given (positive) integer a and all sufficiently large n.

In the sequel, we shall refer to this as Schinzel's conjecture, and show in this paper that this conjecture holds for all $a < 40$. Previously, G. Palama [2], J. Seldlaček [3], and B.M. Stewart and W.A. Webb [6] showed that the conjecture holds for all $a \leq 35$. However, they worked out the details of proof for only some values of a; they did not indicate a systematic approach to the solution, as we believe we do in this note. (See, for example, Theorem 3.5 below.)

In Section 2, we analyze the equation

$$\frac{a}{n} = \frac{1}{x} \pm \frac{1}{y} , \tag{1.3}$$

giving those a for which (1.3) has solutions for all $n \geq a$ and characterizing those n for which (1.3) does not have solutions, especially for small values of a.

In Section 3, we discuss equation (1.2) and use the results of Section 2, namely Lemma 2.1 and Theorem 2.5 to prove its solvability for all $a < 40$, giving a full and systematic account of the various cases that arise. We may mention at this point that while

Lemma 2.1 is an established tool for those working on this problem, Theorem 2.5 and its importance in discussing (1.2) does not seem to have been noticed before.

In Section 4, we consider some general conjectures concerning the number of divisors for integers in the neighborhood of a large integer n and show how (1.1) and (1.2) would become solvable if certain of these conjectures were true.

Finally; in Section 5 we give bounds k depending on the numerator a which allow the representation of a/n as a sum or difference of k simple fractions.

2. Representation of fractions as sum or difference of two simple fractions.

2.1 Lemma. Let a, n be relatively prime positive integers. Then the equation

$$\frac{a}{n} = \frac{1}{x} \pm \frac{1}{y}$$

has solutions in positive integers x, y if and only if there are two distinct divisors d_1, d_2 (positive or negative) of n so that $d_1 \equiv d_2 \pmod{a}$. (Note that we may assume that $(d_1, d_2) = 1$ since any common factor could be cancelled.)

Proof: This result is known (see, for example, Stewart and Webb [6]). However, for the sake of completeness, we give a proof.

Assume $d_1|n$, $d_2|n$, $d_2 > d_1$ and $d_1 \equiv d_2 \pmod{a}$. Then

$$\frac{1}{d_1} - \frac{1}{d_2} = \frac{d_2 - d_1}{d_1 d_2} = \frac{qa}{d_1 d_2},$$

q being an integer. Hence

$$\frac{1}{(qn/d_2)} - \frac{1}{(qn/d_1)} = \frac{a}{n}, \tag{2.2}$$

so that $x = qn/d_2$, $y = qn/d_1$ gives the desired solution.

Conversely, assume that $x > 0$,

$$\frac{a}{n} - \frac{1}{x} = \frac{ax - n}{xn} = \frac{1}{y}$$

and set $d = (x,n)$, $x = dx_1$, $n = dn_1$. Then

$$\frac{ax_1 - n_1}{dx_1n_1} = \frac{1}{y} \tag{2.3}$$

and

$$(ax_1 - n_1, x_1) = (n_1, x_1) = 1,$$

$$(ax_1 - n_1, n_1) = (ax_1, n_1) = 1.$$

Thus (2.3) implies $ax_1 - n_1 = d_1 | d | n$ and $d_1 \equiv -n_1 \pmod{a}$. Thus if we set $n_1 = -d_2$, then $d_1 \not\equiv d_2$ and we get the necessity of the condition.

2.4. __Theorem__. Equation (1.3) has solutions for every $n \geq a$ if and only if $a = 1,2,3,4$ or 6. For every other positive integer a, there are infinitely many values of n for which (1.3) has no integral solutions.

__Proof__: If $\varphi(a) \leq 2$ and $(a,n) = 1$, then $n \equiv \pm 1 \pmod{a}$ and the conditions of Lemma 2.1 are satisfied by the divisors $1, n$. On the other hand, if $\varphi(a) > 2$, then there exist infinitely many primes p with $p \nmid a$ and $p \not\equiv \pm 1 \pmod{a}$, and the conditions of Lemma 2.1 are violated when n equals one of these primes. (cf. Theorem 3 of [6]).

2.5. Theorem. If $\varphi(a) < 2d(n)$, where $d(n)$ is the number of divisors of n, then (1.3) has integral solutions.

Proof: Since the number of distinct d, $d|n$, (positive or negative) exceeds the number of reduced residues (mod a), there must exist distinct divisors d_1, d_2 of n with $d_1 \equiv d_2 \pmod{a}$.

2.6. Corollary. Equation (1.3) has no solutions for the given values of $\varphi(a)$ only under the following conditions on n. Here p, q, r stand for distinct primes.

$\varphi(a)$	n
$2 < \varphi(a) < 6$	$n = p$
$6 \leq \varphi(a) < 8$	$n = p^\alpha$, $\alpha = 1$ or 2
$8 \leq \varphi(a) < 12$	$n = p^\alpha$, $\alpha = 1,2,3$
	$n = pq$
$12 \leq \varphi(a) < 16$	$n = p^\alpha$, $\alpha = 1,2,\ldots,6$
	$n = p^\alpha q$, $\alpha = 1,2$
$16 \leq \varphi(a) < 24$	$n = p^\alpha$, $\alpha = 1,2,\ldots,10$
	$n = p^\alpha q$, $\alpha = 1,2,\ldots,5$
	$n = p^2 q^2$
	$n = p\,q\,r$

3. Verification of Schinzel's conjecture for small values of the numerator.

We first observe that it suffices to prove that (1.2) has solutions for all sufficiently large prime values of n. Assume that (1.2) has solutions for all primes $n \geq N$; then every integer $n \geq N^{\varphi(a)/2}$ which has all of its prime factors $< N$

satisfies $2d(n) > \varphi(a)$, and thus by Theorem 2.5, every such fraction can be expressed as the sum or difference of two simple fractions. Hence we assume that n is a prime, $n \geq 2a$, and write $n = am \pm r$ where $r \leq a/2$, $m \geq 2$.

In view of Theorem 2.4, we also assume that $a \geq 5$. Now

$$\frac{a}{n} = \frac{1}{m} \mp \frac{r}{mn} \tag{3.1}$$

and more generally,

$$\frac{a}{n} = \frac{1}{m \pm s} \pm \frac{as \mp r}{(m \pm s)n}, \quad s = 0, \pm 1, \pm 2, \ldots \tag{3.2}$$

3.3. <u>Theorem</u>. Equation (1.2), with $n \geq 2a$, n prime, has integral solution x, y, z where $1/x$ is the simple fraction nearest to a/n whenever $\varphi(r) < 8$; that is, for all $r \in R = \{1,2,3,4,5,6,7,8,9,10,12,14,18\}$. Thus such a solution exists for all $a \leq 22$.

<u>Proof</u>: Since $1 < m < n$ and n is prime, we have $d(mn) \geq 4$, and by Theorem 2.5,

$$\frac{r}{mn} = \frac{1}{y} \pm \frac{1}{z}$$

has integral solutions whenever $\varphi(r) < 8$. Substituting in (3.1), we get the desired solution of (1.2).

If $a \leq 22$, then $r < 10$ and hence $r \in R$.

Since m is large when n is large, the argument at the beginning of this section shows that we may restrict attention to those m which have some large prime divisor. For our purpose, it suffices to assume that $m \neq 2^{\alpha}3^{\beta}$.

3.4 <u>Theorem</u>. If n is sufficiently large and a, r are both odd, $\varphi(r) < 16$, that is $r \in R = \{11,13,15,21\}$, then (1.2) has an integral solution x, y, z, where $1/x$ is the simple fraction nearest to a/n. Thus such a solution exists whenever $a = 23, 25, 27, 29, 31$ and n is sufficiently large.

<u>Proof</u>: If a and r are both odd, then m is even. Since we have $m \neq 2^{\alpha}$, we get $d(mn) \geq 8$, and by Theorem 2.5,

$$\frac{r}{mn} = \frac{1}{y} \pm \frac{1}{z}$$

has integral solutions whenever $\varphi(r) < 16$. Substituting in (3.1) we get the desired solution of (1.2).

For even $a \leq 31$, the only possible values of r which are not included in R are 11, 13, 15.

In the cases not treated in Theorems 3.3 and 3.4 we will in general not be able to solve (1.2) with $x = m$. We will therefore have to use (3.2) with values of $s \neq 0$.

We proceed to consider some outstanding cases successively using Corollary 2.6.

(i) <u>$r = 11$</u>, a even. If $d(mn) > 4$, we are finished; so we assume m prime and $m \pm 1$ even, so $d((m \pm 1)n) > 8$ and the choice $s = 1$,

$$\frac{a}{n} = \frac{1}{m \pm 1} \pm \frac{a - 11}{(m \pm 1)n}$$

leads to a solution when $\varphi(a - 11) < 16$. This holds for $a = 24, 26$. Unless $m \pm 1 = 2p$, p prime, we have $d((m \pm 1)n) \geq 12$ and we get a solution for $\varphi(a - 11) < 24$. This holds for $a = 28, 30, 32, 34, 36, 38$. If m and $(m \pm 1)/2$ are both primes, then $(m \mp 1)$ is divisible by 12 and the choice $s = -1$,

$$\frac{a}{n} = \frac{1}{m \mp 1} \mp \frac{a + 11}{(m \mp 1)n}$$

with $d((m \mp 1)n) \geq 24$ leads to a solution whenever $\varphi(a + 11) < 48$. This holds for $a = 28, 30, 32, 34, 36, 38, 40$.

Thus the case $r = 11$ is settled for all integers $a < 40$, and since this was the only remaining case for $a = 24, 26$, the conjecture is settled for these values.

(ii) $\underline{r = 13}$, a even. If $d(mn) > 6$, we are finished, so we assume that m is a prime or the square of a prime and have $m \pm 1$ even, so that $d((m \pm 1)n) \geq 8$.

Thus, as before, we get a solution if $\varphi(a - 13) < 16$. This holds for $a = 28, 34$. Unless $m \pm 1 = 2p$, p prime, we have $d((m \pm 1)n) \geq 12$ and we get a solution for $\varphi(a - 13) < 24$. This holds for $a = 30, 32, 36, 38, 40, 46$.

If m and $(m \pm 1)/2$ are relatively prime to 6, then $m \mp 1$ is divisible by 12 and $d((m \mp 1)n) \geq 24$. Thus, as before we get a solution when $\varphi(a + 13) < 48$. This holds for 30, 32, 36, 38.

Thus the case $r = 13$ is settled for all integers $a \leq 38$. Since this was the only remaining case for $a = 28, 30$, the conjecture is settled for these values.

(iii) $\underline{r = 15}$, a even. If $d(mn) > 4$, we are finished, so we again assume m prime, $m \pm 1$ even. Unless $m \pm 1 = 2p$ we are finished when $\varphi(a - 15) < 24$. This holds for $a = 32, 34, 38$. If $m \pm 1 = 2p$, then $m \mp 1$ is divisible by 12 and $d((m \mp 1)n) \geq 24$, so that we are finished when $\varphi(a + 15) < 48$. This holds for $a = 32, 34$.

Thus the case $r = 15$ is settled for $a = 32, 34$. Since this was the only remaining case for $a = 32, 34$, the conjecture is settled for these values.

(iv) $\underline{r = 16}$. If $d(mn) > 4$, we are finished, so m is prime, $m \pm 1$ even, $d((m \pm 1)n) \geq 8$ and we are finished when $\varphi(a-16) < 16$.

This holds for $a = 37$. Unless $m \pm 1 = 2p$, we are finished when $\varphi(a - 16) < 24$. This holds for $a = 33, 35, 39, 41, 43$. If $m \pm 1 = 2p$, then $m \mp 1$ is divisible by 12 and we are finished when $\varphi(a + 16) < 48$. This holds for $a = 33, 35, 39, 41$.

The case $r = 16$ is settled for all integers ≤ 41. Since this was the only remaining case for $a = 33$, the conjecture is settled in this case.

(v) $\underline{r = 17}$. Here we look directly at

$$\frac{a}{n} = \frac{1}{m \pm 1} \pm \frac{a - 17}{(m \pm 1)n} .$$

Hence if $\varphi(a - 17) < 8$, we are finished. This holds for $a = 35$.

Summing up, the conjecture is settled for all values of $a \leq 35$.

We now consider the cases $a = 36, 37, 38, 39$ not settled by the enumeration for the values of r given above.

(vi) $\underline{\text{The case } a = 36, r = 17}$. This is the only case that remains to be checked to settle the conjecture for $a = 36$.

We are finished unless

$$d(m) \leq \frac{\varphi(17)}{4} = 4$$

$$d(m \pm 1) \leq \left[\frac{\varphi(19)}{4}\right] = 4$$

$$d(m \mp 1) \le \frac{\varphi(53)}{4} = 13$$

$$d(m \mp 2) \le \frac{\varphi(55)}{4} = 10 \ .$$

We first prove that we are finished unless $3 \nmid m$. If $m = 3p$, then

$$n \equiv 2m \equiv 6p \pmod{17} \ ;$$

hence

$$\frac{36}{36m \pm 17} = \frac{1}{m} \mp \frac{17}{3pn}$$

and $3n \equiv p \pmod{17}$. So $17/(3pn)$ is the sum of two simple fractions.

We next prove that we are finished unless $3 \nmid (m \pm 1)$. Now if $m \pm 1 = 3q$, we have $n \equiv 17(m \pm 1) \equiv -6q \pmod{19}$,

$$\frac{36}{36m \pm 17} = \frac{1}{m \pm 1} \pm \frac{19}{3qn}$$

and $3n \equiv q \pmod{19}$. Thus $19/(3qn)$ is the sum of two simple fractions. Thus we may assume that $3 | (m \mp 1)$ and $3 | (m \pm 2)$.

If $2|m$, then $m = 2p$, p prime and $4|(m \pm 2)$, but then $12|(m \pm 2)$ and $d(m \pm 2) \ge 2d(12) = 12$, so that we are finished. We may therefore assume $m \pm 1 = 2q$, q prime and

$$m \mp 1 = 12s \ .$$

Now write

$$n \equiv -17(m \mp 1) \equiv 8s \pmod{53}$$

and

$$\frac{36}{36n \pm 17} = \frac{1}{m \mp 1} \pm \frac{53}{12sn} \ ,$$

where the divisors of $12sn$ are represented (mod 53) by $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm s, \pm 2s, \pm 3s, \pm 4s, \pm 6s, \pm 12s, \pm 8s, \pm 16s, \pm 24s, \pm 21s, \pm 5s, \pm 10s, \pm 8s^2, \pm 16s^2, \pm 24s^2, \pm 21s^2, \pm 5s^2, \pm 10s^2$.

If $s \equiv \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 10, \pm 11, \pm 12, \pm 13, \pm 14, \pm 15, \pm 16, \pm 17, \pm 18, \pm 19, \pm 20, \pm 21, \pm 22, \pm 23, \pm 25, \pm 26 \pmod{53}$, then one of the multiples of s is congruent to a divisor of 12 and we are finished.

This leaves the cases $s \equiv \pm 8, \pm 24 \pmod{53}$ now $s^2 \equiv 11, -7 \pmod{53}$. But then $5(\pm 8)^2 \equiv 2 \pmod{53}$ and $8(\pm 24)^2 \equiv -3 \pmod{53}$, and we are again finished - and Schinzel's conjecture holds for $a = 36$.

(vii) a = 37. In view of the earlier results, we need only consider $r = 17$. Since $n = 37m \pm 17$ is an odd prime, we know that m is even. We are finished unless

$$d(m) \leq \varphi(17)/4 = 4 ,$$

that is, unless $m = 2p$, p prime. Now assume $m = 2p$. Since

$$\frac{37}{n} = \frac{1}{m \pm 1} \pm \frac{20}{(m \pm 1)n} ,$$

we are finished unless $d(m \pm 1) \leq \varphi(20)/4 = 2$. Hence $m \pm 1 = q$ is a prime.

Now assume $m = 2p$, $m \pm 1 = q$; then $m \mp 1$ is a multiple of 3 and

$$\frac{37}{n} = \frac{1}{m \mp 1} \mp \frac{54}{(m \mp 1)n} = \frac{1}{m \mp 1} \mp \frac{18}{\left(\frac{m \mp 1}{3}\right)n} ,$$

and $d((m \mp 1)/3) \geq 2 > \varphi(18)/4$. So the last fraction is the sum or difference of two simple fractions by Lemma 2.1.

(viii) $a = 38$. By the above list, we need only consider the cases $r = 15$ and $r = 17$. For $r = 15$, we have

$$\frac{38}{38m \pm 15} = \frac{1}{m} \mp \frac{15}{mn} .$$

We are finished unless $d(m) \leq \varphi(15)/4 = 2$, that is, unless $m = p$, a prime.

Now assume $m = p$. Since

$$\frac{38}{n} = \frac{1}{m \pm 1} \pm \frac{23}{(m \pm 1)n} ,$$

we are finished unless $d(m \mp 1) \leq \varphi(23)/4 = 5.5$, that is, unless $m \pm 1 = 2q$, q prime.

Now assume $m = p$, $m \pm 1 = 2q$. We have

$$\frac{38}{n} = \frac{1}{m \mp 1} \mp \frac{53}{(m \mp 1)n}$$

and

$$m \mp 1 = 12s \ , \quad n \equiv -15\ (m \mp 1) \equiv -21s \pmod{53} .$$

The residue classes (mod 53) of the divisors of $(m \mp 1)n$ are therefore represented by $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm s, \pm 2s, \pm 3s, \pm 4s, \pm 6s, \pm 12s, \pm 21s, \pm 11s, \pm 10s, \pm 22, \pm 20s, \pm 13s, \pm 21s^2, \pm 11s^2, \pm 10s^2, \pm 22s^2, \pm 20s^2, 13s^2$. If $s \equiv \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 9, \pm 10, \pm 11, \pm 12, \pm 14, \pm 15, \pm 16, \pm 17, \pm 18, \pm 19, \pm 20, \pm 21, \pm 23, \pm 24, \pm 25, \pm 26$, then one of the multiples of s is congruent to a divisor of 12. This leaves the cases $s \equiv +7, \pm 13, \pm 22$; $s^2 \equiv 4, 10, 7$, or $13 s^2 \equiv 1$, $10 s^2 \equiv -6$, $21 s^2 \equiv -12$. Thus in all cases, $53/(m \mp 1)n$ is a sum or difference of two simple fractions by Lemma 2.1. In the case $r = 17$, we are finished unless

$$d(m) \leq \varphi(17)/4 = 4 ,$$

that is, $m = p, p^2, p^3$ or pq with p,q primes. We are again finished unless

$$d(m \pm 1) \leq \varphi((38 - 17)/4) = 3 ,$$

that is, $m \pm 1$ a prime or the square of a prime. Thus m is even as the above analysis shows, $m = 2p$, p prime. Thus

$$\frac{1}{n} = \frac{1}{2p} \mp \frac{17}{2pn}$$

where

$$n = 38m \pm 17 \equiv 76p \equiv 8p \pmod{17} .$$

Thus $p \equiv -2n \pmod{17}$ and $17/(2pn)$ is the sum of two simple fractions by Lemma 2.1.

(ix) <u>$a = 39$</u>. We need only consider the cases $r = 17$ and $r = 19$. If $r = 17$, we have $n = 39m \pm 17$ and

$$\frac{39}{n} = \frac{1}{m} \mp \frac{17}{mn} .$$

We are therefore finished unless $d(m) \leq \frac{\varphi(17)}{4} = 4$, and since m is even, this is possible only when $m = 2p$, p prime. Now

$$\frac{39}{n} = \frac{1}{m \pm 1} \pm \frac{22}{(m \pm 1)n} .$$

We are therefore finished unless

$$d(m \pm 1) \leq \frac{\varphi(22)}{4} = 2.5 ,$$

that is, $d(m \pm 1) \leq 2$, and hence $m \pm 1$ is a prime.

If m satisfies these two conditions, then $m \pm 2$ is divisable by 12, say $m \pm 2 = 12s$ and

$$\frac{39}{n} = \frac{1}{m \pm 2} \pm \frac{61}{(m \pm 2)n}$$

where $2n = 78m \pm 34 \equiv 17(m \pm 2) \equiv 17 \cdot 12s \pmod{61}$ so that $n \equiv -20s \pmod{61}$.

Therefore the denominator $(m \pm 2)n$ has two divisors s and $3n$ with $s \equiv 3n \pmod{61}$, and the fraction $61/(m \pm 2)n$ can be expressed as the difference of two simple fractions.

If $r = 19$ we have $n = 39m \pm 19$

$$\frac{39}{n} = \frac{1}{m} \mp \frac{19}{mn}$$

and $m \equiv n \pmod{19}$. So the last fraction is the sum of two simple fractions. Combining all the above results, we have the following.

3.5 Theorem. Equation (1.2) has solutions for all $a < 40$ and sufficiently large n. In the cases $a \leq 35$, the fraction $1/x$ can be chosen among the three nearest neighbors of a/n.

4. Some conjectures and their applications.

4.1 Conjecture. $\lim_{n \to \infty} \sup_{s > 0} \frac{d(n+s)}{s+1} = \infty$.

It is clear that $\sup_s d(n+s)/(s+1)$ is attained for some $s = o(n^{\varepsilon})$. The conjecture says that every sufficiently large integer n has a "successor" n' with many more divisors than the distance $|n' - n|$.

Remark. Conjecture (4.1) implies the truth of the Schinzel - Sierpinski conjecture (which is the strongest conjecture stated in Section 1).

To see this, we write

$$n = am - r , \quad 0 < r < a ,$$

and

$$\frac{a}{n} = \frac{1}{m+s} + \frac{sa+r}{(m+s)n} , \quad s = 0,1,2,\ldots .$$

Conjecture (4.1) implies

$$d(m+s) > a(s+1) > \varphi(as+r)$$

for some $s \geq 0$ when n, and hence m, is sufficiently large. Thus $m + s$ has two divisors d_1, d_2 with $d_1 \equiv -d_2 \pmod{sa + r}$ and it follows from Lemma (2.1) that

$$\frac{sa+r}{(m+s)n} = \frac{1}{y} + \frac{1}{z}$$

has integral solutions.

For Schinzel's equation (1.2), we need the weaker conjecture:

4.3 Conjecture. $\lim_{n \to \infty} \sup_{s \geq 0} \frac{d(n \pm s)}{s + 1} = \infty$.

In order to prove the Schinzel-Sierpinski conjectures, for specific sets A of numerators a, we need the weaker form:

4.4 Conjecture. $\lim_{n \to \infty} \sup_{s \geq 0} \frac{d(n + s)}{\varphi(as - r)} \geq 1$

for all $0 < r < a$, $(r,a) = 1$ and all $a \in A$; and

4.5 Conjecture. $\lim_{n \to \infty} \sup_{s \geq 0} \frac{d(n \pm s)}{\varphi(as \pm r)} > \frac{1}{2}$

for all $0 < r < \frac{a}{2}$, $(r,a) = 1$ and all $a \in A$.

Remark. As before, Conjecture 4.4 implies the Schinzel-Sierpinski conjecture and Conjecture 4.5 implies the Schinzel conjecture.

We remark that P. Erdös has conjectured a strong negation of Conjecture 4.1, namely,

4.6 Conjecture. There exist arbitrarily large n for which $d(n + s)/(s + 1) \leq 2$ for all $s = 0,1,2,\ldots$. This would be possible only if n is prime, $n + 1$ twice a prime, $n + 2$ a prime or thrice a prime or nine times a prime or thrice a prime square, etc.

In connection with the above conjectures the following question appears to be of great interest in itself.

4.7 Question. For what functions $f(s)$ do we have $d(n + s) \leq f(s)$, $s = 0, \pm 1, \pm 2,\ldots$, and infinitely many n? P. Erdös has given us an argument using sieve methods to show that such functions $f(s)$ do in fact exist.

It would also be interesting to get asymptotic quantitative estimates. That is, for Conjectures 4.1, 4.3 and 4.6 we would like to know an estimate for the number of integers $n \leq x$ for which the quantities $\sup_{s \geq 0} d(n \pm s)/(s + 1)$ are less than a given constant y. Similarly for Conjectures 4.4 and 4.5 we would like to estimate the number of integers $n \leq x$ for which the conjectured inequalities fail for any given a and r.

5. Some remarks. We may ask the more modest question: what bound $k = k(a)$ can be given so that the equation

$$\text{(5.1)} \qquad \frac{a}{m} = \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_k}$$

has solutions in (not necessarily positive) integers $x_1, \dots, x_k$ for all sufficiently large n? We can prove the following.

5.2 Theorem. $k \leq \frac{1}{2} \log_2 (3(a + 2))$.

Proof. If, as before, we write $n = am \pm r$ with $r \leq a/2$ and set $x_1 = m$, we get

$$\frac{a}{n} = \frac{1}{m} \mp \frac{r}{mn} .$$

Thus, in trying to reduce the size of the numerator by as much as possible, we write $m = rm_1 \pm r_1$, $n = rn_1 \pm r_1'$, $mn = rs_1 \pm r_1''$ with $1 \leq r_1, r_1', r_1'' < r/2$. If two of r, r', r'' are equal, then $k \leq 3$, otherwise

$$\min\{r_1, r_1', r_1''\} \leq \frac{r}{2} - 3 .$$

choosing the smallest remainder available in each case, we get

$$\frac{a}{m} = \frac{1}{x_1} + \ldots + \frac{1}{x_{\ell-1}} + \frac{r_\ell}{n_\ell}$$

where

$$d(n_\ell) \geq 2^{\ell-1}$$

and either $\frac{r_\ell}{n_\ell}$ is the sum of two simple fractions or

$$r_\ell \leq \frac{r_{\ell-1}}{2} - 2^\ell + 1 .$$

Iterating the inequality for r_k, we get

$$r_\ell \leq \frac{a}{2^\ell} - 2^\ell(1 + \frac{1}{4} + \ldots) + (1 + \frac{1}{2} + \ldots)$$

$$\leq \frac{a}{2^\ell} - \frac{4}{3} \cdot 2^\ell + 2 .$$

We know that r_ℓ/n_ℓ is the sum or difference of two simple fractions once

$$\varphi(r_\ell) < 2\, d(n_\ell) < 2^{\ell+2} .$$

In particular, if $r_\ell < 2^{\ell+2}$ we get $k \leq \ell + 1$. Combining (5.1) and (5.2) we have $r_\ell \leq 2^{\ell+2}$ whenever

$$\frac{a}{2^\ell} \leq 2^{\ell+2} + \frac{1}{3} 2^{\ell+2} - 2 ,$$

or when $2^{2\ell+4} \geq 3(a + 2)$, that is

$$\ell \geq \frac{1}{2} \log_2[3(a + 2)] - 2 .$$

we thus have

$$k = k(a) \leq [\frac{1}{2} \log_2(3(a + 2))] .$$

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