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TRANSFORMATION FORMULAE FOR MULTIPLE SERIES

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In this paper we prove a general transformation formula for a triple series of complex terms. We deduce a transformation formula for double series and discuss some applications. Furthermore, some reciprocity relations of the following type are obtained: Let a, b, c be reals greater than 1 and

$$P(a,b,c)=\sum_{r=1}^{\infty} r^{-a}\sum_{k=1}^{r} k^{-b}\sum_{l=1}^{k} l^{-c}.$$

Then

$$\begin{aligned}P(a,b,c)+P(a,c,b)+P(b,c,a)+P(b,a,c)&\\ +P(c,a,b)+P(c,b,a)&\\ =\zeta(a)\zeta(b)\zeta(c)+\zeta(a)\zeta(b+c)&\\ +\zeta(b)\zeta(c+a)+\zeta(c)\zeta(a+b)+2\zeta(a+b+c),&\end{aligned}$$

where ζ denotes the Riemann zeta function. In particular, $P(2,2,2)=31\pi^6/15{,}120$.

1. Introduction. Since the time of Euler, the evaluation of certain infinite series, in closed form, in terms of the Riemann zeta function and allied functions is familiar. The processes involved, at times, yield recurrence relations among these functions. Results of this kind, dating back to 1743 and due to Euler, can be found in N. Nielsen's book (cf. **[6]**, Erster Teil, Kapitel III). It appears that some recent authors are not aware of these results. For example, in 1953, G. T. Williams (cf. **[9]**, Theorems III and I) proved the following results:

$$(1.1)\quad 2\sum_{r=1}^{\infty}\frac{1}{r^a}\sum_{k=1}^{r}\frac{1}{k}=(a+2)\zeta(a+1)-\sum_{i=1}^{a-2}\zeta(a-i)\zeta(i+1),$$

$$(1.2)\quad \begin{aligned}&\zeta(2)\zeta(2a-2)+\zeta(4)\zeta(2a-4)+\cdots+\zeta(2a-2)\zeta(2)\\ &\quad=\left(a+\tfrac{1}{2}\right)\zeta(2a),\end{aligned}$$

where a is an integer ≥ 2 and ζ denotes the Riemann zeta function defined by $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$ for $s>1$. Williams claims that (1.1) is "apparently entirely new" and that the convolution considered in (1.2) "seems never to have been explicitly formulated before" (see also **[3]**, **[4]**, **[2]**, and **[7]**). But we note that (1.1) and (1.2) are already deduced in

Nielsen (cf. [6], p. 47, Eqs. (3) and (4)) in a very elegant way and apparently go back to Euler (see footnotes on p. 47 of Nielsen [6]). Probably Nielsen's techniques were essentially those of Euler!. Nielsen (cf. [6], p. 47) makes use of some reciprocity relations to evaluate certain infinite series and further to deduce various recurrence relations involving the Riemann zera function and allied functions ((1.1) and (1.2) are just typical of these results). A careful examination of Nielsen's proofs shows that they hinge on a certain transformation formula for double series which, however, is not explicitly stated in the book. In §2 we prove a transformation formula of a very general nature for triple series, which is believed to be new. In §3 we deduce a transformation formula for double series, which in turn yields as special cases all the basic reciprocity relations on p. 47 of Nielsen's book [6]. In §4 we further illustrate our transformation formula for triple series by deducing various reciprocity relations and also evaluate some infinite series as special cases. Typical of our results is the following:

$$\sum_{r=1}^{\infty} \frac{1}{r^2} \sum_{k=1}^{r} \frac{1}{k^2} \sum_{l=1}^{k} \frac{1}{l^2} = \frac{31}{15{,}120}\pi^6.$$

2. Transformation formula for triple series. Let $\sum_{r,k,l=1}^{\infty} f(r,k,l)$ be an absolutely convergent triple series with complex terms. Then the following transformation formula holds.

THEOREM 2.1. *We have*

$$\begin{aligned}
&\sum_{r,k,l=1}^{\infty} f(r,k,l) \\
&\quad = \sum_{r=1}^{\infty} \sum_{k=1}^{r} \sum_{l=1}^{k} \{f(r,k,l) + f(r,l,k) + f(k,l,r) \\
&\qquad\qquad + f(k,r,l) + f(l,r,k) = f(l,k,r)\} \\
&\quad - \sum_{r=1}^{\infty} \sum_{k=1}^{r} \{f(r,k,k) + f(r,k,r) + f(k,r,k) + f(k,r,r)\} \\
&\quad - \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} f(r,r,k).
\end{aligned}$$

Proof. The absolute convergence of the triple series $\sum_{r,k,l=1}^{\infty} f(r,k,l)$ justifies the rearrangements that we perform in the sequel. We write

$$\begin{aligned}
(2.1)\qquad \sum_{r,k,l=1}^{\infty} f(r,k,l) &= \sum_{r=1}^{\infty} \sum_{k\le r} \sum_{l=1}^{\infty} f(r,k,l) + \sum_{r=1}^{\infty} \sum_{k>r} \sum_{l=1}^{\infty} f(r,k,l) \\
&= \Sigma_1 + \Sigma_2
\end{aligned}$$

and

$$(2.2)\qquad \sum 1 = \sum_{r=1}^{\infty} \sum_{k\le r} \sum_{l\le k} f(r,k,l) + \sum_{r=1}^{\infty} \sum_{k\le r} \sum_{l>k} f(r,k,l) = \sum\nolimits_1^{(1)} + \sum\nolimits_1^{(2)}.$$

We have

$$\begin{aligned}
\sum\nolimits_1^{(2)} &= \sum_{r=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k\le r, k<l} f(r,k,l) \\
&= \sum_{r=1}^{\infty} \sum_{2\le l\le r} \sum_{k\le r, k<l} f(r,k,l) + \sum_{r=1}^{\infty} \sum_{l>r} \sum_{k\le r, k<l} f(r,k,l) \\
&= \sum_{r=1}^{\infty} \sum_{2\le l\le r} \Big(\sum_{k\le l} f(r,k,l) - f(r,l,l) \Big) + \sum_{l=2}^{\infty} \sum_{r<l} \sum_{k\le r} f(r,k,l) \\
&= \sum_{r=1}^{\infty} \sum_{l\le r} \Big(\sum_{k<l} f(r,k,l) - f(r,l,l) \Big) \\
&\quad + \sum_{l=2}^{\infty} \Big(\sum_{r\le l} \sum_{k\le r} f(r,k,l) - \sum_{k\le l} f(l,k,l) \Big) \\
&= \sum_{r=1}^{\infty} \sum_{l\le r} \sum_{k\le l} f(r,k,l) - \sum_{r=1}^{\infty} \sum_{l\le r} f(r,l,l) \\
&\quad + \sum_{l=1}^{\infty} \sum_{r\le l} \sum_{k\le r} f(r,k,l) - \sum_{l=1}^{\infty} \sum_{k\le l} f(l,k,l).
\end{aligned}$$

Thus from (2.1) and (2.2), we obtain

$$\begin{aligned}
(2.3)\qquad \sum\nolimits_1 &:= \sum_{r=1}^{\infty} \sum_{k\le r} \sum_{l=1}^{\infty} f(r,k,l) \\
&= \sum_{r=1}^{\infty} \sum_{k\le r} \sum_{l\le k} f(r,k,l) + \sum_{r=1}^{\infty} \sum_{l\le r} \sum_{k\le l} f(r,k,l) \\
&\quad - \sum_{r=1}^{\infty} \sum_{l\le r} f(r,l,l) + \sum_{l=1}^{\infty} \sum_{r\le l} \sum_{k\le r} f(r,k,l) \\
&\quad - \sum_{l=1}^{\infty} \sum_{k\le l} f(l,k,l) \\
&= \sum_{r=1}^{\infty} \sum_{k\le r} \sum_{l\le k} \{ f(r,k,l) + f(r,l,k) + f(k,l,r) \} \\
&\quad - \sum_{r=1}^{\infty} \sum_{k\le r} \{ f(r,k,k) + f(r,k,r) \}.
\end{aligned}$$

Now we consider Σ_2. We also have

$$\begin{aligned}\Sigma_2 &= \sum_{r=1}^{\infty} \sum_{k>r} \sum_{l=1\infty} f(r,k,l) = \sum_{k=2}^{\infty} \sum_{r<k} \sum_{l=1}^{\infty} f(r,k,l) \\ &= \sum_{k=2}^{\infty} \left(\sum_{r\le k} \sum_{l=1}^{\infty} f(r,k,l) - \sum_{l=1}^{\infty} f(k,k,l) \right) \\ &= \sum_{k=1}^{\infty} \sum_{r\le k} \sum_{l=1}^{\infty} f(r,k,l) - \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} f(k,k,l).\end{aligned}$$

We observe that the first sum on the right of above is similar to that considered in Σ_1. Hence by (2.3) and the above, we obtain

$$\begin{aligned}(2.4)\quad \Sigma_2 = &\sum_{r=1}^{\infty} \sum_{k\le r} \sum_{l\le k} \{(k,r,l) + f(l,r,k) + f(l,k,r)\} \\ &- \sum_{r=1}^{\infty} \sum_{k\le r} \{f(k,r,k) + f(k,r,r)\} - \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} f(r,r,k).\end{aligned}$$

Finally, on combining (2.1), (2.3) and (2.4), we arrive at

$$\begin{aligned}&\sum_{r,k,l=1}^{\infty} f(r,k,l) \\ &\quad = \sum_{r=1}^{\infty} \sum_{k\le r} \sum_{l\le k} \{f(r,k,l) + f(r,l,k) + f(k,l,r) \\ &\qquad\qquad + f(k,r,l) + f(l,r,k) + f(l,k,r)\} \\ &\quad - \sum_{r=1}^{\infty} \sum_{k\le r} \{f(r,k,k) + f(r,k,r) + f(k,r,k) + f(k,r,r)\} \\ &\quad - \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} f(r,r,k).\end{aligned}$$

This completes the proof of Theorem 2.1.

3. A transformation formula for double series. Let $\Sigma_{r,k=1}^{\infty} g(r,k)$ be an absolutely convergent double series with complex terms. Then the following transformation formula holds.

THEOREM 3.1. *We have*

$$\sum_{r,k=1}^{\infty} g(r,k) = \sum_{r=1}^{\infty} \sum_{k\le r} g(r,k) + \sum_{r=1}^{\infty} \sum_{k\le r} g(k,r) - \sum_{r=1}^{\infty} g(r,r).$$

Proof. On taking

$$f(r,k,l)=\begin{cases} g(r,k) & \text{if } l=1,\\ 0 & \text{otherwise,}\end{cases}$$

in Theorem 2.1, we obtain the assertion.

REMARK 3.1. A direct and simple proof of Theorem 3.1 could be given as follows:

$$\begin{aligned}\sum_{r,k=1}^{\infty} g(r,k) &= \sum_{r=1}^{\infty}\sum_{k\le r} g(r,k)+\sum_{r=1}^{\infty}\sum_{k>r} g(r,k)\\ &= \sum_{r=1}^{\infty}\sum_{k\le r} g(r,k)+\sum_{k=2}^{\infty}\sum_{r<k} g(r,k)\\ &= \sum_{r=1}^{\infty}\sum_{k\le r} g(r,k)+\sum_{k=1}^{\infty}\Big(\sum_{r\le k} g(r,k)-g(k,k)\Big)\\ &= \sum_{r=1}^{\infty}\sum_{k\le r} g(r,k)+\sum_{r=1}^{\infty}\sum_{k\le r} g(k,r)-\sum_{r=1}^{\infty} g(r,r).\end{aligned}$$

The following special case of Theorem 2.1 will be particularly useful in the applications we had in mind.

THEOREM 3.2. *Let* $\sum_{r=1}^{\infty} f(r)$ *and* $\sum_{k=1}^{\infty} g(k)$ *be two absolutely convergent series of complex terms. Then*

$$\left(\sum_{r=1}^{\infty} f(r)\right)\left(\sum_{k=1}^{\infty} g(k)\right)=\sum_{r=1}^{\infty} f(r)\sum_{k\le r} g(k)+\sum_{r=1}^{\infty} g(r)\sum_{k\le r} f(k)-\sum_{r=1}^{\infty} f(r)g(r). \tag{3.1}$$

In particular,

$$2\sum_{r=1}^{\infty} f(r)\sum_{k\le r} f(k)=\left(\sum_{r=1}^{\infty} f(r)\right)^2+\sum_{r=1}^{\infty} f^2(r). \tag{3.2}$$

The following reciprocity theorem is an easy consequence of (3.1).

THEOREM 3.3. *For* $i\in\{0,1\}$, *define the function* x_i *by setting*

$$x_i(n)=(-1)^{i(n-1)} \quad \textit{for positive integral } n$$

and write, for $s > 1$, $f_i(s) = \sum_{n=1}^{\infty} x_i(n) n^{-s}$. For reals a, b greater than 1 and $i, j \in \{0, 1\}$, let

$$P_{i,j}(a, b) = \sum_{r=1}^{\infty} \frac{x_i(r)}{r^a} \sum_{k \le r} \frac{x_j(k)}{k^b}.$$

Then

$$P_{i,j}(a, b) + P_{j,i}(b, a) = f_i(a) f_j(b) + f_{i+j}(a + b).$$

REMARK 3.2. We note that for $i \in \{0, 1\}$,

$$f_i(s) = (1 - i 2^{1-s}) \zeta(s),$$

and, furthermore, Theorem 3.3 contains as special cases all the basic reciprocity relations, in a slightly different notation, on p. 47 of Nielsen [**6**]. As alluded to in the introduction, Nielsen made use of these relations to evaluate some infinite series and prove several recurrence relations and (1.1) and (1.2) are representatives of these. It may be of interest to note that M. S. Klamkin [**3**] posed the special case $a = 3$ of (1.1) as a problem, and while submitting his solution [**4**] attributed (1.1) to G. T. Williams. Unaware of Nielsen's and William's work, W. E. Griggs, S. Chowla, A. J. Kempner and W. E. Mientka [**2**], proved (1.1) in the case $a = 2$ and also that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left\{ \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \right\} = \zeta(3). \tag{3.3}$$

But interestingly (1.1) in the case $a = 2$ and (3.3) are in fact equivalent in view of the following: For any $a > 1$ we have

$$\sum_{r=1}^{\infty} \frac{1}{r^a} \sum_{k \le r} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{r} \sum_{k > r} \frac{1}{k^a} + \zeta(a + 1).$$

This follows from

$$\begin{aligned}\sum_{r=1}^{\infty} \frac{1}{r} \sum_{k > r} \frac{1}{k^a} &= \sum_{k=2}^{\infty} \frac{1}{k^a} \sum_{r < k} \frac{1}{r} = \sum_{k=2}^{\infty} \frac{1}{k^a} \left(\sum_{r \le k} \frac{1}{k} - \frac{1}{k} \right) \\ &= \sum_{k=1}^{\infty} \frac{1}{k^a} \left(\sum_{r \le k} \frac{1}{r} - \frac{1}{k} \right) = \sum_{k=1}^{\infty} \frac{1}{k^a} \sum_{r \le k} \frac{1}{r} - \zeta(a + 1).\end{aligned}$$

Alternative proofs of (1.1) and (1.2) could be found, respectively, in R. Sita Ramachandra Rao and A. Siva Rama Sarma [**7**] and A. Siva Rama Sarma [**8**]. Their arguments are based on a certain generalization of a transformation formula due to J. Lehner and M. Newman [**5**].

4. Further applications of Theorem 2.1. Let u, v, $w \in \{0, 1\}$ and a, b, c be reals greater than 1. We write

$$P_{u,v,w}(a, b, c) = \sum_{r=1}^{\infty} \frac{x_u(r)}{r^a} \sum_{k \le r} \frac{x_v(k)}{k^b} \sum_{l \le k} \frac{x_2(l)}{l^c}$$

where the function x_i is as defined in §3. Then the following reciprocity relation holds.

THEOREM 4.1. *We have*

$$\begin{aligned} P_{u,v,w}(a, b, c) + P_{u,w,v}(a, c, b) + P_{v,w,u}(b, c, a) + P_{v,u,w}(b, a, c) \\ + P_{w,u,v}(c, a,) + P_{w,v,u}(c, b, a) \\ = f_u(a)f_v(b)f_w(c) + f_u(a)f_{v+w}(b+c) + f_v(b)f_{w+u}(c+a) \\ + f_w(c)f_{u+v}(a+b) + 2f_{u+v+w}(a+b+c) \end{aligned}$$

where $f_i(s)$ is as given in Theorem 3.3.

Proof. On taking $f(r, k, l) = x_u(r)r^{-a}x_v(k)k^{-b}x^w(l)l^{-c}$ in Theorem 2.1, we obtain

$$\begin{aligned} (4.1) \quad & f_u(a)f_v(b)f_w(c) \\ & = P_{u,v,w}(a, b, c) + P_{u,w,v}(a, c, b) + P_{v,w,u}(b, c, a) \\ & \quad + P_{v,u,w}(b, a, c) + P_{w,u,v}(c, a, b) + P_{w,v,u}(c, b, a) \\ & \quad - \{P_{u,v+w}(a, b+c) + P_{v+w,u}(b+c, a) \\ & \qquad + P_{uw,v}(a+c, b) + P_{v,u+w}(b, a+c)\} \\ & \quad - f_{u+v}(a+b)f_w(c). \end{aligned}$$

But, by Theorem 3.3, we have

$$\begin{aligned} & P_{u,v+w}(a, b+c) + P_{v+w,u}(b+c, a) \\ & \quad = f_u(a)f_{v+w}(b+c) + f_{u+v+w}(a+b+c), \\ & P_{u+w,v}(a+c, b) + P_{v,u+w}(b, a+c) \\ & \quad = f_{u+w}(a+c)f_v(b) + f_{u+v+w}(a+b+c). \end{aligned}$$

On substituting these on the right of (4.1), we arrive at the assertion of the Theorem.

REMARK 4.1. It may be noted that on choosing u, v, w from among 0 and 1, the above reciprocity theorem yields eight distinct reciprocity relations. For example, writing

$$P(a, b, c) = P_{0,0,0}(a, b, c) \quad \text{and} \quad P_1(a, b, c) = P_{1,1,1}(a, b, c),$$

we have

$$(4.2)\quad \begin{aligned} P(a,b,c) + P(a,c,b) + P(b,c,a)j + P(b,a,c) \\ +P(c,a,b) + P(c,b,a) \\ = \zeta(a)\zeta(b)\zeta(c) + \zeta(a)\zeta(b+c) + \zeta(b)\zeta(c+a) \\ +\zeta(c)\zeta(a+b) + 2\zeta(a+b+c), \end{aligned}$$

$$(4.3)\quad \begin{aligned} P_1(a,b,c) + P_1(a,c,b) + P_1(b,c,a) + P_1(b,a,c) \\ +P_1(c,a,b) + P_1(c,b,a) \\ = (1-2^{1-a})(1-2^{1-b})(1-2^{1-c})\zeta(a)\zeta(b)\zeta(c) \\ + (1-2^{1-a})\zeta(a)\zeta(b+c) + (1-2^{1-b})\zeta(b)\zeta(c+a) \\ + (1-2^{1-c})\zeta(c)\zeta(a+b) + 2(1-2^{1-(a+b+c)})\zeta(a+b+c). \end{aligned}$$

Specializing (4.2) and (4.3) with $a = b = c$, we obtain

$$(4.4)\quad \begin{aligned} P(a,a,a) := \sum_{r=1}^{\infty} r^{-a} \sum_{k<r} k^{-a} \sum_{l<k} l^{-a} \\ = \tfrac{1}{6}\{\zeta^3(a) + 3\zeta(a)\zeta(2a) + 2\zeta(3a)\}, \end{aligned}$$

$$(4.5)\quad \begin{aligned} P_1(a,a,a) := \sum_{r=1}^{\infty} (-1)^{r-1} r^{-a} \sum_{k\le r} (-1)^{k-1} k^{-a} \sum_{l\le k} (-1) l^{-1} l^{-a} \\ = \tfrac{1}{6}\Big\{(1-2^{1-a})^3\zeta^3(a) + 3(1-2^{1-a})\zeta(a)\zeta(2a) \\ +2(1-2^{1-3a})\zeta(3a)\Big\}. \end{aligned}$$

Since it is well known that for positive integral n, $\zeta(2n)$ is a rational multiple of π^{2n} (cf. [**1**], §12.12), from (4.4) and (4.5), we conclude that if a is an even integer ≥ 2, then each of $P(a,a,a)$ and $P_1(a,a,a)$ is a rational multiple of π^{3a}. Further, since $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ and $\zeta(6) = \pi^6/945$ (cf. [**1**], §12.12), we have, in particular,

$$\sum_{r=1}^{\infty} r^{-2} \sum_{k\le r} k^{-2} \sum_{l\le k} l^{-2} = \frac{31}{15{,}120}\pi^6,$$

$$\sum_{r=1}^{\infty} (-1)^{r-1} r^{-2} \sum_{k\le r} (-1)^{k-1} \sum_{l\le k} (-1)^{l-1} l^{-2} = \frac{109}{120{,}960}\pi^6.$$

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