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A REMARK ON A PAPER WRITTEN BY J. M. DE KONINCK AND A. IVIĆ

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Abstract: Let $\mathcal{F}$ denote the set of monotonically decreasing functions f defined on $(0,\infty)$ such that $f(x)\to 0$ $(x\to\infty)$. Let $q_1<\ldots<q_r$ be the prime divisors of n, $\tau(n)$ = number of divisors of n, $d_1<d_2<\ldots<d_{\tau(n)}$ be the sequence of divisors of n. Let $f\in\mathcal{F}$, and

$$h(n)=\sum_{i=1}^{r-1} f(q_{i+1}-q_i);\quad H(n):=\sum_{i=1}^{\tau(n)-1} f(d_{i+1}-d_i),$$

$$A(x)=\sum_{n\le x} h(n);\qquad B(x)=\sum_{n\le x} H(n).$$

It is proved:
a) $A(x)/x$ tends to a finite limit if and only if $\sum h^{-1}f(2^h)<\infty$. If the series converges then h is almost periodic in Besicovitch sense,
b) $B(x)/x$ has a finite limit if and only if $\sum f(2^h)<\infty$. If the series converges then H is almost periodic in Besicovitch sense.

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1. Let $n = q_1^{\alpha_1} \dots q_r^{\alpha_r}$, $q_1 < \dots < q_r$ be the prime divisors of n, $\tau(n)$ be the divisor function. Let $d_1 < d_2 < \dots < d_{\tau(n)}$ be the sequence of positive divisors of n. We shall say that a function f belongs to $\mathcal{F}$, if it is defined for $x \geq 1$ and tends to zero monotonically as $x \to \infty$, $f(x) > 0$. For a fixed $f \in \mathcal{F}$ let the functions $h : \mathbb{N} \to \mathbb{R}$, $H : \mathbb{N} \to \to \mathbb{R}$ be defined as follows: $h(1) = 0$, $H(1) = 0$, $h(p^\alpha) = 0$ if p^α is a prime-power, otherwise

$$h(n) = \sum_{i=1}^{r-1} f(q_{i+1} - q_i); \qquad H(n) := \sum_{i=1}^{\tau(n)-1} f(d_{i+1} - d_i). \tag{1.1}$$

Let

$$A(x) = \sum_{n \leq x} h(n); \qquad B(x) = \sum_{n \leq x} H(n). \tag{1.2}$$

In their paper [1] J. M. De Koninck and A. Ivić proved that for $f(x) = 1/x$ the relations

$$A(x) = Ax + O\left(\frac{x \log\log x}{\log x}\right), \quad B(x) = Bx + O\left(\frac{x}{(\log x)^{1/3}}\right)$$

hold with suitable explicitly given constants A, B.

Our aim is to generalize these results for large classes of functions $f \in \mathcal{F}$. (We will not take care about the remainder terms.)

Since $f(q_{i+1} - q_i) \geq f(q_{i+1})$ if $f \in \mathcal{F}$, therefore

$$A(x) \geq \sum_{2 < p < x^{1/4}} f(p) \sum_{2mp \leq x} 1 \geq \frac{x}{2} \sum_{2 < p \leq x^{1/4}} \frac{f(p)}{p},$$

where in the sum p runs over the set of primes.

So for the existence of the mean value of h the condition

$$\sum f(p) p^{-1} < \infty, \tag{1.3}$$

i.e. the equivalent condition

$$\sum_{h=1}^{\infty} f(2^h) h^{-1} < \infty \tag{1.4}$$

is necessary.

Since $f(d_{i+1} - d_i) \geq f(d_{i+1})$, we get similarly that $B(x) \geq \geq \sum f(d) \left[\frac{x}{d}\right]$. If $B(x) = O(x)$, then

$$\sum_{d=1}^{\infty} d^{-1} f(d) < \infty, \tag{1.5}$$

i.e.

$$\sum_{h=1}^{\infty} f(2^h) < \infty \tag{1.6}$$

holds.

It is much more interesting that the conditions (1.4), (1.5) are sufficient for the existence of the mean value for h, H, respectively.

2. Let us consider h under (1.4). For an $y \geq 2$ let P_y be the product of the primes not greater than y. For an $n \in \mathbb{N}$ let $A_y(n)$ be the product of the prime divisors of n not greater than y. Then $n = A_y(n)B_y(n)$, $(B_y(n),\ P_y) = 1$.

Let $s_y(n) := h(A_y(n))$. Since $h(n)$ does not depend on the multiplicity of the prime divisors of n, therefore $s_y(n) = s_y(\ell)$ if $n \equiv \ell (\operatorname{mod} P_y^2)$, i.e. $s_y(n)$ is periodic mod P_y^2.

Let $t_y(n) := h(n) - s_y(n)$. Then

$$t_y(n) = h(B_y(n)) + f(v(n) - u(n))$$

where $v(n)$ is the smallest prime divisor of $B_y(n)$ and $u(n)$ is the largest prime divisor of $A_y(n)$. Consider the sum

$$\sum\nolimits_1 := \sum_{n \leq x} f(v(n) - u(n)).$$

The number of the integers $n \leq x$ satisfying $u(n) > y/2$ is less than

$$\sum_{\frac{y}{2} < p \leq y} \sum x/p \ll \frac{x}{\log y}.$$

Furthermore, if $u(n) \leq y/2$, then $f(v(n) - u(n)) \leq f(y/2)$, consequently

$$\sum\nolimits_1 \ll x f(y/2) + \frac{cx}{\log y}.$$

Let

$$\sum\nolimits_2 := \sum_{n \leq x} h(B_y(n)).$$

Then

$$\sum\nolimits_2 = \sum_{y<q\le x} \sum_{y<p<q} f(q-p)D_{p,q},$$

where $D_{p,q}$ denotes the number of those $n \le x$ for which p and q are consecutive prime divisors. Let $\sum_2 = \sum_{2,1} + \sum_{2,2}$, where in $\sum_{2,1}$ $\frac{q}{2} \le$ $\le p < q$, while in $\sum_{2,2}$ $p < q/2$. To estimate $\sum_{2,1}$ we shall use the crude estimation $D_{p,q} \le x/pq$, and we get

$$\sum\nolimits_{2,1} \ll x \sum_{y<q<x} \frac{1}{q} \left(\sum_{q/2<p<q} 1/p \right) \ll x \sum_{y<q} \frac{1}{q \log q},$$

from the convergence of $\sum (q \log q)^{-1}$ we get that $\sum_{2,1} = o_y(1)x$ $(x \to$ $\to \infty)$. To estimate $\sum_{2,2}$ we observe that $f(p-q) \le f(q/2)$ and that $\sum_p D_{p,q} \le x/q$. So by (1.4) we have

$$\sum\nolimits_{2,2} \le \sum_{y<q\le x} f(q/2)[x/q] = o_y(1)x.$$

Collecting our results we get

$$\lim_{y\to\infty} \left(\limsup \frac{1}{x} \sum_{n\le x} |h(n) - s_y(n)| \right) = 0. \tag{2.1}$$

Since $s_y(n)$ is periodic mod P_y^2, therefore (2.1) implies that h is almost periodic, consequently the mean value of h exists. For the proof of this conclusion see [2], Ch. 4, Th. 4. As a consequence we get that h has a limit distribution, i.e.

$$\lim \frac{1}{x} \# \{h(n) < \lambda,\ n \le x\} = V(\lambda) \tag{2.2}$$

exists for almost all real number λ.

So we proved

Theorem 1. *Let $f \in \mathcal{F}$. Then $A(x)x^{-1}$ tends to a finite limit if and only if (1.4) holds. If (1.4) holds, then h is almost periodic in Besicovitch sense.*

3. Let us consider now H. Assume that (1.5) holds, and that $f \in \mathcal{F}$. It is obvious that

$$\sum_{n=2}^{\infty} n^{-1} f\left(n/(\log n)^3\right) < \infty. \tag{3.1}$$

Indeed, let us sum $1/n$ over those n for which $2^h < \dfrac{n}{(\log n)^3} < 2^{h+1}$. The smallest n is $A_h \geq c_1 2^h h^3$, the largest n is $B_h \leq c_2 2^h \cdot h^3$, where c_1, c_2 are positive constants. Then

$$\sum_{A_h \leq n \leq B_h} 1/n \leq \log \frac{c_2}{c_1}.$$

Hence (3.1) immediately follows.

Let $\rho(z) = z(\log z)^{-1}$,

$$t_y(n) := \sum_{d_i > \rho(y)} f(d_{i+1} - d_i), \qquad T_y(x) = \sum_{n \leq x} t_y(n).$$

Now we estimate $T_y(x)$. The contribution of the terms in $T_y(x)$ satisfying $d_{i+1} - d_i > \rho(d_i)$ is less than

$$x \sum_{d > \rho(y)} f(\rho(d)) 1/d = o_y(1) x.$$

Now we consider the terms for which $d_{i+1} - d_i < \rho(d_i)$. If $d_i,\ d_{i+1}$ are consecutive divisors of n such that $\sqrt{n} < d_i < d_{i+1}$, then n/d_{i+1}, n/d_i are consecutive divisors of n as well, furthermore $0 < \dfrac{n}{d_i} - \dfrac{n}{d_{i+1}} < d_{i+1} - d_i$. Consequently in the estimation of $T_y(x)$ the terms $d_i > \sqrt{n}$ can be cancelled.

Let us consider the contribution of those pairs $d_i,\ d_{i+1}$ for which $d_i < \sqrt{n} < d_{i+1}$. Then $d_{i+1} = \dfrac{n}{d_i}$.

Since $0 < d_{i+1} - d_i < \rho(d_i)$, for a fixed d_i the number of distinct d_{i+1} is less than $O\left(\rho(d_i)\right)$, consequently the contribution of them is less than

$$\sum_{d < \sqrt{x}} \frac{d}{\log d} = o(x) \qquad (x \to \infty).$$

It has remained to estimate the sum

$$\sum\nolimits_1 := \sum_m \sum_\ell f(\ell) \sum_n 1,$$

where m and ℓ run over the intervals $1 \leq \ell \leq \rho(m)$, $\rho(y) \leq m \leq \sqrt{x}$, $m + \ell \leq \sqrt{x}$, while $n \leq x$ and $m|n$, $m + \ell|n$. The innermost sum is $\leq \dfrac{x}{[m, m+\ell]}$. Let $\rho = (m, m, +\ell) = (m, \ell)$, $m = \delta m^*$, $\ell = \delta m^*$. Then we get

$$\sum\nolimits_1 \leq x \sum_m \sum_{\delta/m} \sum_{s \leq \frac{\rho(m)}{\delta}} \frac{f(\delta s)}{\delta m^*(m^*+s)} \leq x \sum_m \frac{1}{m} \sum_{\delta, s} \frac{f(\delta s)}{m^*},$$

where in the inner sum $\delta | m$, $s\delta \leq \rho(m)$.

Summing for m in a subinterval of type $[M, 2M]$, we get

$$\sum_{M \leq m \leq 2M} \frac{1}{m} \sum_{s,\delta} \frac{f(\delta s)}{m^*} \leq \frac{1}{m} \sum_{k \leq \rho(2M)} f(k) \sum_{\delta/k} \left(\sum 1/s\right),$$

where in the innermost sum in the right-hand side s runs over the interval $\left[\dfrac{M}{\delta}, \dfrac{2M}{\delta}\right]$. So the right-hand side is less than

$$\frac{1}{M} \sum_{k \leq \rho(2M)} f(k)\tau(k).$$

Let now $M_1 = 2$, $M_{\nu+1} = 2M_\nu$ and we sum over those M_t for which $M_{t+1} \geq \rho(y)$. Then we get

$$\sum\nolimits_1 \ll x \sum_{M_t} \frac{1}{M_t} \sum_{k \leq \rho(M_{t+1})} f(k)\tau(k). \tag{3.2}$$

Let us change the summation in the right-hand side. The least value of M_t for which k occurs in the inner sum is of order $k \log k$. Since $\sum\limits_{t \geq t_0} M_t^{-1} \ll M_{t_0}^{-1}$, we get that

$$\sum\nolimits_1 \ll x \sum_k f(k)\tau(k) \min\left(\frac{1}{\rho(y)}, \frac{1}{k \log k}\right). \tag{3.3}$$

Now we prove that

$$\sum_k \frac{f(k)\tau(k)}{k \log k} < \infty \tag{3.4}$$

holds.

Let us sum (3.4) for $k \in [2^h,\ 2^{h+1}]$. The sum is less than

$$\frac{f(2^h)}{2^h \log 2^h} \sum_{k < 2^{h+1}} \tau(k) \ll \frac{f(2^h)}{h},$$

and (3.4) can be overestimated by the convergent sum $\sum h^{-1} f(2^h)$.

Collecting our results we get that $T_y(x) = o_y(1)x$ as $x \to \infty$.

Now we deduce that H is an almost periodic function. For a prime $p \le y$ let $\partial_p(y)$ be the least integer for which $p^{\partial_p(y)} \ge y$. Let $Q_q = \prod_{p \le y} p^{\partial_p(y)}$, furthermore

$$q_y(n) = \sum_{d_{i+1} \le y} f(d_{i+1} - d_i).$$

It is obvious that $n \equiv \ell (\bmod Q_y)$ implies that $q_y(n) = q_y(\ell)$ so q_y is periodic mod Q_y. Furthermore $0 \le H(n) \le t_y(n) + R(n)$, where $R(n) = 0$ except when the largest divisor d of n which is not greater than y is $\le y/\log y$. In this second case $R(n) = f(k-d)$ where k is the smallest divisor of n that is greater than y. So we have $R(n) \le$ $\le f(y - y/\log y) \le f(y/2)$. Consequently $R(n) = o_y(1)$ uniformly as $n \le x$. Taking into account that $T_y(x) = o_y(1)x$ $(x \to \infty)$, we get immediately that

$$\lim_{y \to \infty} \left(\limsup \frac{1}{x} \sum_{n \le x} |H(n) - q_y(n)| \right) = 0. \tag{3.5}$$

(3.5) means that H is almost periodic. Consequently the following assertion holds true.

Theorem 2. *Let $f \in \mathcal{F}$. Then $B(x)x^{-1}$ has a finite limit if and only if (1.6) holds. If (1.6) holds then $H(n)$ is almost periodic in Besicovitch sense.*

References

[1] DE KONINCK, J. M. and IVIĆ, A.: On the distance between consecutive divisors of an integer, *Canad. Math. Bull.* **29** (2) (1986), 208–217.

[2] POSTNIKOV, A. G.: Introduction into analytical number theory, Nauka, 1971 (in Russian).