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On a Function Connected with $\varphi(n)$

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1. Let $\varphi(n)$ denote Euler's function representing the number of numbers less than and prime to n. Let $\varphi_1(n) = \varphi(n)$; $\varphi_r(n) = \varphi(\varphi_{r-1}(n))$, $r = 2, 3, \ldots$. For a given n, $\varphi_r(n)$ decreases as r increases, and hence there is a least value of r, say r_1, such that

$$\varphi_r(n) = \varphi_{r+1}(n) = \ldots = 1.$$

S. S. Pillai proved that

$$\left[\frac{\log (n/2)}{\log 3}\right] + 1 \leqslant R(n) \leqslant \frac{\log n}{\log 2} + 1.$$

where $r_1 = R(n)$. In this paper I consider an analogous function $S(n)$ given by

$$S(n) = \varphi_1(n) + \varphi_2(n) + \ldots + \varphi_{r1}(n).$$

It is shown here that

$$R(n) \leqslant \frac{\log n}{\log 2} \text{ if } n \text{ is even}$$

and

$$\leqslant \frac{\log (n-1)}{\log 2} + 1 \text{ if } n \text{ is odd,}$$

which is an improvement over Pillai's result;

Also

$$S(n) \leqslant n - 1 \text{ if } n \text{ is even}$$

$$\leqslant 2n - 3 \quad \text{,,} \quad \text{odd,}$$

while $S(n) \geqslant 2^{[\log (n/2)/\log 3]+1} - 1$.

I also consider the solutions of the equation $S(n) = n$, and show that each one of the following values of n provides a solution :

$$n = n_1, 3n_2, 3n_3, \ldots.$$

where $n_1 = 3^k,\ k = 0, 1, 2, \ldots.$

$$n_2 = 1 + 4n_1$$
$$n_3 = 1 + 12n_2$$
$$n_4 = 1 + 12n_3, \ldots.$$

provided these are primes. The question whether these are the only primes remains open.

2. *Theorem* 1. If n is even,

$$R(n) \leqslant \frac{\log n}{\log 2};\ S(n) \leqslant n - 1.$$

$$\text{If } n \text{ is odd, } R(n) \leqslant \frac{\log (n-1)}{\log 2} + 1$$

$$S(n) \leqslant 2n - 3.$$

Proof: When n is even, $\varphi_1(n) \leqslant (n/2)$;

$$\varphi_2(n) \leqslant \frac{1}{2}\varphi_1(n) \leqslant \frac{1}{4}n; \ldots.$$

$$\varphi_r(n) \leqslant \frac{1}{2^r}n.$$

If $r = r_1 = R(n)$ we get

$$1 \leqslant \frac{1}{2^{r_1}}n,$$

$$r_1 \leqslant \frac{\log n}{\log 2} \quad \ldots (1)$$

If n is a power of 2, r_1 is actually $\left(\frac{\log n}{\log 2}\right)$

Again $S(n) = \varphi_1(n) + \varphi_2(n) + \ldots + \varphi_{r1}(n)$

$$\leqslant \frac{1}{2}n + \frac{1}{4}n + \ldots + \frac{1}{2^{r_1}}n;$$

$$= n\left(1 - \frac{1}{2^{r_1}}\right)$$

$$\leqslant n\left(1 - \frac{1}{n}\right), \text{ by } (1)$$

Hence $S(n) \leqslant n - 1$.

If n is a power of 2, it is easily seen that $S(n) = n - 1$.

Let n be odd. Then $\varphi_2(n) \leqslant \frac{1}{2}\varphi_1(n)$;

$$\varphi_3(n) \leqslant \frac{1}{2}\varphi_2(n) \leqslant \frac{1}{2^2}\varphi_1(n); \ldots.$$

$$\varphi_r(n) \leqslant \frac{1}{2^{r-1}}\varphi_1(n)$$

Hence $$1 \leqslant \frac{1}{2^{r_1-1}}\varphi(n),$$

$$r_1 \leqslant \frac{\log \varphi(n)}{\log 2} + 1 \leqslant \frac{\log (n-1)}{\log 2} + 1 \qquad .. \quad (2)$$

If n is a prime of the form $2^k + 1$, we get the equality sign. Again

$$\begin{aligned} S(n) &\leqslant \varphi(n)\,(1 + \tfrac{1}{2} + \ldots + (1/2^{r_1-1})) \\ &= 2\varphi(n)\,(1 - (1/2^{r_1})) \\ &\leqslant 2(n-1)\,(1 - (1/2^{r_1})) \\ &\leqslant 2(n-1)\,(1 - (1/2n-1)), \text{ by } (2) \\ &= 2n - 3. \end{aligned}$$

Actually $S(n)$ attains this value if n is a prime of the form $2^k + 1$.

3. Before considering the lower bounds of $R(n)$ and $S(n)$ we prove the following:

Theorem 2.* For a given value of $R(n)$, the maximum value of n is $2 \cdot 3^{R(n)-1}$.

The proof of this depends on the

Lemma. $R(pn) > R(n) + (\log n/\log p)$, if p is a prime > 3. If $p = 3$ it becomes an equality.

Proof by induction: Suppose the result is true for all primes $q < p$ so that

$$R(qn) \geqslant R(n) + \frac{\log q}{\log 3} \qquad .. \quad (3)$$

* This is equivalent to Pillai's theorem II; but the proof by induction given here is different from Pillai's.

for all $n > 0$ and all $q < p$. We will prove the same for p. Let $p - 1 = 2^a q_1^{b_1} q_2^{b_2} \ldots.$

Then $R(n) = 1 + R[\varphi(n)]$ .. (4)

assuming as we may that $n > 2$; for if $n = 2$, the result of the lemma is obvious.

Also $R(pn) = 1 + R(\varphi(pn))$.

Assume that p is prime to n as a first case. Then

$$\begin{aligned} R(pn) &= 1 + R[(p-1)\varphi(n)] \\ &= 1 + R[2^a q_1^{b_1} q_2^{b_2} \ldots \varphi(n)] \\ &= 1 + R[q_1^{b_1} q_2^{b_2} \ldots \varphi(n)] + a, \end{aligned}$$

since $\varphi(n) \geqslant 2$ and $R(2m) = R(m) + 1$ if m is even;

Hence $R(pn) \geqslant 1 + R[\varphi(n)] + a + \Sigma \left(\frac{\log qi}{\log 3}\right) bi,$

by using (3) repeatedly,

$$= R(n) + a + \left\{\frac{\log (p-1)/2^a}{\log 3}\right\}$$

$$= R(n) + \frac{\log (3/2)^a (p-1)}{\log 3}$$

$$\geqslant R(n) + \left(\frac{\log p}{\log 3}\right)$$

if $(3/2)^a(p-1) \geqslant p$ which is true since $a \geqslant 1$, and $p \geqslant 3$. Since the result is true for $p = 3$, the lemma follows in the case when p is prime to n. Next let $(p, n) \neq 1$. Then in each of the pairs $(\varphi_1(pn); \varphi_1(n))$; $(\varphi_2(pn); \varphi_2(n))$; .... p occurs upto a stage, say in the first k pairs, and in the pair $(\varphi_{k+1}(pn); \varphi_{k+1}(n))$ the first member contains p while the second does not.

Let us call $\varphi_{k+1}(np) = pu$; $\varphi_{k+1}(n) = u$; then $(p, u) = 1$. Hence by the above result,

$$R(pu) > R(u) + \left(\frac{\log p}{\log 3}\right) \quad .. \quad (4)$$

But

$$\mathrm{R}(pn) = k + 1 + \mathrm{R}(pu)$$
$$\mathrm{R}(n) = k + 1 + \mathrm{R}(u)$$

Hence adding $k+1$ to both sides of (A) we get

$$\mathrm{R}(pn) > \mathrm{R}(n) + \left(\frac{\log p}{\log 3}\right), \; p > 3,$$

so that the lemma follows in this case also. To prove Theorem 2, let $\mathrm{R}(n)=t$ (fixed), and consider all possible values of n satisfying this. The greatest of such n's, say n_1, must not contain a prime factor p greater than 3, for then

$$\mathrm{R}(n_1) = \mathrm{R}\left(\frac{n_1}{p}\cdot p\right) \geqslant \mathrm{R}\left(\frac{n_1}{p}\right) + \left(\frac{\log p}{\log 3}\right)$$
$$\geqslant \mathrm{R}\left(\frac{n_1}{p}\right) + \left[\frac{\log p}{\log 3}\right] + 1$$
$$= \mathrm{R}\left(\frac{n_1}{p}\cdot 3^k\right),$$

where $\mathrm{K} = \left[\frac{\log p}{\log 3}\right] + 1$, since $\mathrm{R}(3u) = \mathrm{R}(u) + 1$ for all u.

But $\frac{n_1}{p}\cdot 3^k > n_1$. Hence n_1 is not the largest solution.

Let now $n_1 = 3u$; then $t = \mathrm{R}(3u) = \mathrm{R}(u) + 1$ so that $\mathrm{R}(u) = t - 1$. Assuming that the theorem is true for all values of $\mathrm{R}(n) < t$, it follows that the maximum value of u is 2.3^{t-2}. Hence $3u = 2.\,3^{t-1}$ and the theorem follows by induction. From the theorem it follows, as proved by Pillai, that

$$\mathrm{R}(n) \geqslant \left[\frac{\log n - \log 2}{\log 3}\right] + 1. \qquad \text{.. (5)}$$

Theorem 3. $\mathrm{S}(n) \geqslant [\log\ (n/2)/\log\ 3] + 1$.

For $\mathrm{S}(n) = \varphi_1(n) + \varphi_2(n) + \ldots + \varphi_t(n)$; $t = \mathrm{R}(n)$

Now $\varphi_t(n) = 1$; $\varphi_{t-1}(n) = 2$; $\varphi_{t-2}(n) \geqslant 2^2$; $\varphi_{t-3}(n) \geqslant 2^3$; ...

Hence $\mathrm{S}(n) \geqslant 2^{t-1} + 2^{t-2} + \ldots + 2 + 1$

$= 2^t - 1.$

The theorem now follows by using (5).

(4) *$S(n)$ as a function of $R(n)$.* We will show that

$$2^{R(n)} - 1 \leqslant S(n) \leqslant 3^{R(n)}.$$

For a given $R(n) = t$, proceeding as above we get $S(n) \geqslant 2^t - 1 = 2^{R(n)} - 1$. When n is a power of 2, this becomes an equality. Again, by Theorem 2,

$$\varphi_t(n) = 1;\ \varphi_{t-1}(n) = 2;\ \varphi_{t-2}(n) \leqslant 2\cdot 3;\ \varphi_{t-3}(n) \leqslant 2\cdot 3^2 \ldots.$$
$$\varphi_1(n) \leqslant 2\cdot 3^{t-1}$$

Hence for a given t,

$$S(n) \leqslant 2\cdot 3^{t-1} + 2\cdot 3^{t-2} + \ldots + 2\cdot 3 + 2 + 1$$
$$= 3^t.$$

$S(n)$ actually reaches this value when $n = 2\cdot 3^{t-1}$.

4. Let us finally consider the solutions of the equation $S(n) = n$. By Theorem 1, n can only be odd. It is easily verified that two sets of solutions are $n = 3^k$ $(k = 0, 1, \ldots)$ and $n = 3p$ where p is a prime $= 4$. $3^k + 1$, $k = 1, 2, \ldots$; we will prove

Teorem 4. If $S(3p) = 3p$ then $S((p-1)/4) = ((p-1)/4)$, where p is a prime ≥ 3.

Proof. $3p = S(3p) = 2(p-1) + S(2(p-1))$

$$= 2(p-1) + 2S(p-1) + 1,$$

using $S(2u) = 2S(u) + 1$ if u is even

$$= S(u) \qquad \text{if } u \text{ is odd.}$$

Hence $S(p-1) = \dfrac{p+1}{2}$

Since $p > 3$, $\dfrac{p-1}{2}$ is even,

Hence $S\left(\dfrac{p-1}{2}\right) = \left(\dfrac{p-1}{4}\right).$

Since $S(n)$ is always odd, $\frac{p-1}{4}$ must be odd, $p=4k+1$ (k odd);

$\frac{p-1}{2}=2k$, k odd $S\left(\frac{p-1}{2}\right)=S\left(2\cdot\frac{p-1}{4}\right)=S.\left(\frac{p-1}{4}\right)$

$=S$ since $\frac{p-1}{4}$ is odd.

Hence the theorem follows.

With the help of this result, we can derive an infinite sequence of classes of solutions of $S(n)=n$.

One is $n=3p$, where $\frac{p-1}{4}=3^a$ or $p=4\cdot 3^a+1=p_1$ say.

Another is $n=3\ p_2$ where p_2 is a prime given by $\frac{p_2-1}{4}=3\,p_1$

or $p_2=12p_1+1$. .... ..

This sequence of solutions can be continued indefinitely. It is conjectured that this sequence exhausts all solutions.

REFERENCE

S. S. Pillai (1929) .. On a function connected with φ (n): *Bulletin of the American Math. Society, pp.* 837-841.