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Some New Identities Involving the Partition Function $p(n)$

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Here we give some partition identities of a recursive nature. They resemble the well known Euler recursions for the partition function $p(n)$.

1. Introduction

Let $p(n)$ denote as usual, the number of unrestricted partitions of n. Throughout this paper, $\varphi(x)$ denotes the Euler product defined by

$$\varphi(x) = \prod_{n=1}^{\infty}(1-x^n), \quad |x| < 1$$

It is well known, as first proved by Euler, that

$$\varphi(x) = \sum_{-\infty}^{\infty}(-1)^k x^{3k^2+k/2}$$

$$= 1 + \sum_{1}^{\infty}(-1)^k (x^{3k^2+k/2} + x^{3k^2-k/2}) \tag{1}$$

and

$$1/\varphi(x) = \sum_{n=0}^{\infty} p(n)x^n, \quad p(0) = 1. \tag{2}$$

We use the convention that $p(n) = 0$ whenever n is a negative integer.

From the above two relations we get the well known recursion relation for $p(n)$:

$$p(n) = \sum_{k \geq 1} (-1)^k \left\{ p\left(n - \frac{3k^2 - k}{2}\right) + p\left(n - \frac{3k^2 + k}{2}\right) \right\} \tag{3}$$

The above formula has been proven to be useful in investigation of the problem of parity of $p(n)$. For example in 1959 O. Kolberg using (3) proved that $p(n)$ takes both even and odd values, each of them infinitely often. This result is a special case of an old conjecture of M. Newman (1960); which states that for all $m \geq 2$ $p(n) \equiv r \pmod{m}$, $0 \leq r \leq m-1$, has infinitely many solutions in n. It has been proven for $m = 2, 5, 7, 13, 17, 19, 29, 31, 65$ and 121. Also it is a special case of another conjecture of M.V. Subbarao (1966) which says that for all integers $a \geq 1$, each of the congruences: $p(an + b) \equiv 0 \pmod 2$, $p(an + b) \equiv 1 \pmod 2$ has, for each b $(0 \leq b \leq a - 1)$, infinitely many solutions. So far it is known to be true for $a = 1$, 2, 4, 8 and 16. See [1].

In this paper we shall obtain some recursion identities for $p(n)$ which are believed to be new. For this purpose we need to utilize, in addition to the Euler expansion of $\varphi(x)$, the following identities due to Jacobi.

$$\varphi^3(x) = \sum_{k=0}^{\infty} (-1)^k (2k+1) x^{k(k+1)/2}, \qquad |x| < 1 \tag{4}$$

and the Triple Product Identity:

$$\varphi(z^2) \prod_{n=1}^{\infty} (1 + yz^{2n-1})(1 + y^{-1}z^{2n-1}) = \sum_{k=-\infty}^{\infty} y^k z^{k^2} \tag{5}$$

where $|z| < 1$, $y \neq 0$.

2. New Recursion Identities for $p(n)$

Analogous to the Euler recursion formula (3), we shall prove:

Theorem 1.

$$p(2n+1) + \sum_{k>0} \{p(2n+1-(8k^2-2k)) + p(2n+1-(8k^2+2k))\}$$

$$= p(2n) + \sum_{k>0} \{p(2n-(8k^2-6k)) + p(2n-(8k^2+6k))\} \tag{6}$$

$$p(2n)+\sum_{k>0}(-1)^k\{p(2n-(3k^2-k))+p(2n-(3k^2+k))\}$$
$$=p(n)+\sum_{k>0}\{p(n-(4k^2-k))+p(n-(4k^2+k))\} \tag{7}$$

$$p(2n+1)+\sum_{k>0}(-1)^k\{p(2n+1-(3k^2-k))+p(2n+1-(3k^2+k))\}$$
$$=p(n)+\sum_{k>0}\{p(n-(4k^2-3k))+p(n-(4k^2+3k))\} \tag{8}$$

Proof. By applying the Jacobi triple product identity (5) we have:

$$\sum_{-\infty}^{\infty}x^{2n^2-n}=\prod_{n=1}^{\infty}(1-x^{4n})(1+x^{4n-3})(1+x^{4n-1})$$
$$=\prod_{n=1}^{\infty}(1+x^{2n-1})(1-x^{4n})$$
$$=\prod_{n=1}^{\infty}\frac{(1+x^{4n-2})(1-x^{4n})}{1-x^{2n-1}}$$
$$=\prod_{n=1}^{\infty}\frac{1-x^{2n}}{1-x^{2n-1}}$$
$$=\frac{\prod_{n=1}^{\infty}(1-x^{2n})^2}{\prod_{n=1}^{\infty}1-x^{n}}$$
$$=\varphi^2(x^2)\sum_{n=0}^{\infty}p(n)x^n$$
$$=\varphi^2(x^2)\sum_{n=0}^{\infty}p(2n)x^{2n}+\varphi^2(x^2)\sum_{n=0}^{\infty}p(2n+1)x^{2n+1} \tag{9}$$

Now

$$\sum_{n=-\infty}^{\infty}x^{2n^2-n}=\sum_{n=-\infty}^{\infty}x^{8n^2-2n}+\sum_{n=-\infty}^{\infty}x^{8n^2-6n+1} \tag{10}$$

From (9) and (10) we obtain

$$\varphi^2(x^2)\sum_{n=0}^{\infty} p(2n)x^{2n} = \sum_{n=-\infty}^{\infty} x^{8n^2-2n} \tag{11}$$

and

$$\varphi^2(x^2)\sum_{n=0}^{\infty} p(2n+1)x^{2n+1} = \sum_{n=-\infty}^{\infty} x^{8n^2-6n+1} \tag{12}$$

that is

$$\varphi^2(x^2)\sum_{n=0}^{\infty} p(2n+1)x^{2n} = \sum_{n=-\infty}^{\infty} x^{8n^2-6n} \tag{13}$$

To prove (6) we eliminate the $\varphi^2(x^2)$ term from (11) and (13) and equate coefficients of like powers of x.

To prove (7) we rewrite (11) in the form:

$$\varphi(x^2)\sum_{n=0}^{\infty} p(2n)x^{2n} = \frac{1}{\varphi(x^2)}\sum_{k=-\infty}^{\infty} x^{8k^2-2k}$$

and using (1) and (2) we derive:

$$\sum_{k=-\infty}^{\infty}(-1)^k x^{3k^2+k}\sum_{n=0}^{\infty} p(2n)x^{2n} = \sum_{n=0}^{\infty} p(n)x^{2n}\sum_{k=-\infty}^{\infty} x^{8k^2-2k} \tag{14}$$

now (7) follows from (14) on equating coefficients of x. Similarly (8) follows from (13).

3. Further Identities For $p(n)$

Let N be a non-negative integer and let k, l, r, s be integers. Define:

$$a_N = \sum_{48N+5=4(6k+1)^2+(24l+1)^2} (-1)^k ,$$

$$b_N = \sum_{48N+5=4(6r+1)^2+(24s+7)^2} (-1)^r ,$$

$$c_N = \sum_{48N+29=4(6k+1)^2+(24l+19)^2} (-1)^k ,$$

$$d_N = \sum_{48N+29=4(6r+1)^2+(24s+13)^2} (-1)^r ,$$

Then the following holds:

Theorem 2.

$$\sum_{\substack{v \geq 0,\ m \geq 0 \\ \frac{1}{2}v(v+1)+2m=N}} (-1)^v (2v+1) p(4m) + \sum_{\substack{v \geq 0,\ m \geq 0 \\ \frac{1}{2}v(v+1)+2m=N-1}} (-1)^v (2v+1) p(4m+2) = a_N - b_N . \tag{15}$$

$$\sum_{\substack{v \geq 0,\ m \geq 0 \\ \frac{1}{2}v(v+1)+2m=N}} (-1)^v (2v+1) p(4m+1) + \sum_{\substack{v \geq 0,\ m \geq 0 \\ \frac{1}{2}v(v+1)+2m=N-1}} (-1)^v (2v+1) p(4m+3) = c_N - d_N . \tag{16}$$

Proof. We use the formulas (1.5), (1.7) and (3.7) as given in [2]. Thus

$$\sum_0^\infty p(4m) x^{2m} + \sum_0^\infty p(4m+2) x^{2m+1} = \frac{\varphi(x^2)}{\varphi^3(x)} \varphi(x^{24}) A_1(x), \tag{17}$$

where

$$\varphi(x^{24}) A_1(x) = \sum_{-\infty}^{\infty} x^{l(12l+1)} - \sum_{-\infty}^{\infty} x^{(3s+1)(4s+1)} \tag{18}$$

Combining the above formulas we obtain

$$\varphi^3(x) \left\{ \sum_0^\infty p(4m) x^{2m} + \sum_0^\infty p(4m+2) x^{2m+1} \right\}$$

$$= \sum_{k,l=-\infty}^{\infty} (-1)^k x^{3k^2+k+12l^2+l} - \sum_{r,s=-\infty}^{\infty} (-1)^r x^{3r^2+r+12s^2+7s+1}$$

$$= \sum_{N=0}^{\infty} (a_N - b_N) x^N . \tag{19}$$

Now (15) follows on equating the coefficients of like powers of x in (19).

The proof of (16) actually follows the lines of the previous case, but we do not have a formula analogous to (18), but which we now develop. Using the formulas (1.6) and (1.8) in [2] we have:

$$\sum_{0}^{\infty} p(4m+1)x^{2m} + \sum_{0}^{\infty} p(4m+3)x^{2m+1} = \frac{\varphi(x^2)}{\varphi^3(x)}\varphi(x^{24})A_3(x) \tag{20}$$

where

$$A_3(x) = \prod_{m=1}^{\infty}(1+x^{24m-17})(1+x^{24m-7}) - x^2\prod_{m=1}^{\infty}(1+x^{24m-23})(1+x^{24m-1}) \tag{21}$$

(See [2], page 348.) We need to have $\varphi(x^{24})A_3(x)$ expressed in the form of an infinite series. Let us put $y = x^{-5}$, $z = x^{12}$ in (5). Then

$$\varphi(x^{24})\prod_{m=1}^{\infty}(1+x^{24m-17})(1+x^{24m-7}) = \sum_{k=-\infty}^{\infty} x^{12k^2-5k} = \sum_{l=-\infty}^{\infty} x^{(l+1)(12l+7)} \tag{22}$$

Similarly letting $y = x^{-11}$ and $z = x^{12}$ we get

$$x^2\varphi(x^{24})\sum_{m=1}^{\infty}(1+x^{24m-23})(1+x^{24m-1}) = \sum_{k=-\infty}^{\infty} x^{12k^2-11k+2} = \sum_{s=-\infty}^{\infty} x^{(4s+3)(3s+1)}. \tag{23}$$

Combining (20), (21), (22) and (23) we derive

$$\varphi^3(x)\left\{\sum_{m=0}^{\infty} p(4m+1)x^{2m} + \sum_{m=0}^{\infty} p(4m+3)x^{2m+1}\right\} = \varphi(x^2)\left\{\sum_{l=-\infty}^{\infty} x^{(l+1)(12l+7)} - \sum_{m=0}^{\infty} x^{(4s+3)(3s+1)}\right\} \tag{24}$$

Now (16) follows on the lines of the proof of Theorem 2.

Corollary 1. *If* $48N + 5$ *is not a sum of two squares then:*

$$\sum_{\substack{v \geq 0,\ m \geq 0 \\ \frac{1}{2}v(v+1)+2m=N}} p(4m) \quad + \sum_{\substack{v \geq 0,\ m \geq 0 \\ \frac{1}{2}v(v+1)+2m=N-1}} p(4m+3) \equiv 0 \pmod 4 \tag{25}$$

If $48N + 29$ *is not a sum of two squares then:*

$$\sum_{\substack{v \geq 0,\ m \geq 0 \\ \frac{1}{2}v(v+1)+2m=N}} p(4m+1) \quad + \sum_{\substack{v \geq 0,\ m \geq 0 \\ \frac{1}{2}v(v+1)+2m=N-1}} p(4m+3) \equiv 0 \pmod 4 \tag{26}$$

Proof. Since $\varphi^3(x) \equiv \sum_{n=0}^{\infty} x^{n(n+1)/2} \pmod 4$, then (25) and (26) follow immediately from (15) and (16) respectively.

Remark 1. We apply (6) to provide a new proof of Kolberg's result, that is $p(n)$ takes both even and odd values, each of them infinitely often.

Assume first that $p(n) \equiv 0 \pmod 2$ for all $n \geq t$, and let $2n = 8t^2 + 6t$. Then $p(8t^2 + 6t + 1) + \ldots + p(8t + 1) + p(4t + 1) \equiv 0 \pmod 2$ and $p(8t^2 + 6t) + \ldots$ $\ldots + p(12t) + p(0) \equiv 1 \pmod 2$ contradiction.

If $p(n) \equiv 1 \pmod 2$ for all $n \geq t$, then we set $2n = 8t^2 + 2t$ and:

$$p(8t^2 + 2t + 1) + \sum_{0<k\leq t-1} \{p(8t^2 + 2t + 1 - 8k^2 + 2k)$$
$$+ p(8t^2 + 2t + 1 - 8k^2 - 2k)\} + \{p(4t + 1) + p(1)\} \equiv 1 \pmod 2$$

since we have an odd number of odd terms. On the other hand:

$$p(8t^2 + 2t) + \sum_{0<k\leq t-1} \{p(8t^2 + 2t - 8k^2 + 6k)$$
$$+ p(8t^2 + 2t - 8k^2 - 6k)\} + p(8t) \equiv 0 \pmod 2$$

since now, there are an even number of odd terms.

Added in Proof: F. Garavan and D. Stanton (unpublished) have recently verified the Subbarao conjecture for $a = 3,5$ and 10.

References

[1] *M.V. Subbarao,* On the parity of the partition function. Congressus Numeratium **56** (1987), 265—275.

[2] *M.V. Subbarao* and *V.V. Subrahmanyasastri,* A transformation formula for products arising in partition theory. The Rocky Mountain J. Math. Vol. 6 (2), 1976, 345—356.

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