Some New Identities Involving the Partition Function \( p(n) \)

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Here we give some partition identities of a recursive nature. They resemble the well known Euler recursions for the partition function \( p(n) \).

1. Introduction

Let \( p(n) \) denote as usual, the number of unrestricted partitions of \( n \). Throughout this paper, \( \varphi(x) \) denotes the Euler product defined by

\[
\varphi(x) = \prod_{n=1}^{\infty} (1 - x^n), \quad |x| < 1
\]

It is well known, as first proved by Euler, that

\[
\varphi(x) = \sum_{-\infty}^{\infty} (-1)^k x^{3k^2 + k/2} = 1 + \sum_{1}^{\infty} (-1)^k (x^{3k^2 - k/2} + x^{3k^2 - k/2})
\]

and

\[
\frac{1}{\varphi(x)} = \sum_{n=0}^{\infty} p(n)x^n, \quad p(0) = 1.
\]

We use the convention that \( p(n) = 0 \) whenever \( n \) is a negative integer.
From the above two relations we get the well known recursion relation for \( p(n) \):

\[
p(n) = \sum_{k \geq 1} (-1)^k \left\{ p \left( n - \frac{3k^2 - k}{2} \right) + p \left( n - \frac{3k^2 + k}{2} \right) \right\}
\]  

(3)

The above formula has been proven to be useful in investigation of the problem of parity of \( p(n) \). For example in 1959 O. Kolberg using (3) proved that \( p(n) \) takes both even and odd values, each of them infinitely often. This result is a special case of an old conjecture of M. Newman (1960); which states that for all \( m \geq 2 \)

\[ p(n) \equiv r \pmod{m}, \quad 0 \leq r \leq m - 1, \]

has infinitely many solutions in \( n \). It has been proven for \( m = 2, 5, 7, 13, 17, 19, 29, 31, 65 \) and 121. Also it is a special case of another conjecture of M.V. Subbarao (1966) which says that for all integers \( a \geq 1 \), each of the congruences: \( p(an + b) \equiv 0 \pmod{2} \), \( p(an + b) \equiv 1 \pmod{2} \) has, for each \( b \) \( (0 \leq b \leq a - 1) \), infinitely many solutions. So far it is known to be true for \( a = 1, 2, 4, 8 \) and 16. See [1].

In this paper we shall obtain some recursion identities for \( p(n) \) which are believed to be new. For this purpose we need to utilize, in addition to the Euler expansion of \( \phi(x) \), the following identities due to Jacobi.

\[
\frac{\phi^3(x)}{x} = \sum_{k=0}^{\infty} (-1)^k (2k + 1)x^{k(k+1)/2}, \quad |x| < 1
\]  

(4)

and the Triple Product Identity:

\[
\phi(z^2) \prod_{n=1}^{\infty} (1 + yz^{2n - 1})(1 + y^{-1} z^{2n - 1}) = \sum_{k=-\infty}^{\infty} y^k z^k
\]  

(5)

where \( |z| < 1, y \neq 0 \).

2. New Recursion Identities for \( p(n) \)

Analogous to the Euler recursion formula (3), we shall prove:

**Theorem 1.**

\[
p(2n + 1) + \sum_{k>0} \left\{ p \left( 2n + 1 - (8k^2 - 2k) \right) + p \left( 2n + 1 - (8k^2 + 2k) \right) \right\}
\]

\[= p(2n) + \sum_{k>0} \left\{ p \left( 2n - (8k^2 - 6k) \right) + p \left( 2n - (8k^2 + 6k) \right) \right\}
\]  

(6)
\[ p(2n) + \sum_{k > 0} (-1)^k \{ p(2n - (3k^2 - k)) + p(2n - (3k^2 + k)) \} \]
\[ = p(n) + \sum_{k > 0} \{ p(n - (4k^2 - k)) + p(n - (4k^2 + k)) \} \quad (7) \]
\[ p(2n + 1) + \sum_{k > 0} (-1)^k \{ p(2n + 1 - (3k^2 - k)) + p(2n + 1 - (3k^2 + k)) \} \]
\[ = p(n) + \sum_{k > 0} \{ p(n - (4k^2 - 3k)) + p(n - (4k^2 + 3k)) \} \quad (8) \]

**Proof.** By applying the Jacobi triple product identity (5) we have:
\[ \sum_{n=-\infty}^{\infty} x^{2n^2 - n} = \prod_{n=1}^{\infty} \left( 1 - x^{4n} \right) \left( 1 + x^{4n-2} \right) \left( 1 + x^{4n-1} \right) \]
\[ = \prod_{n=1}^{\infty} \left( 1 + x^{2n-1} \right) (1 - x^{4n}) \]
\[ = \prod_{n=1}^{\infty} \left( 1 + x^{4n-2} \right) (1 - x^{4n}) \quad 1 - x^{2n-1} \]
\[ = \prod_{n=1}^{\infty} \frac{1 - x^{2n}}{1 - x^{2n-1}} \]
\[ = \prod_{n=1}^{\infty} \frac{(1 - x^{2n})^2}{1 - x^n} \]
\[ = \frac{\varphi^2(x^2) \sum_{n=0}^{\infty} p(n)x^n}{\prod_{n=1}^{\infty} (1 - x^n)} \]
\[ = \varphi^2(x^2) \sum_{n=0}^{\infty} \frac{p(2n)x^{2n}}{x^n} + \varphi^2(x^2) \sum_{n=0}^{\infty} \frac{p(2n+1)x^{2n+1}}{x^n} \quad (9) \]

Now
\[ \sum_{n=-\infty}^{\infty} x^{2n^2 - n} = \sum_{n=-\infty}^{\infty} 8n^2 - 2n + 1 + \sum_{n=-\infty}^{\infty} 8n^2 - 6n + 1 \quad (10) \]
From (9) and (10) we obtain
\begin{equation}
\varphi^2 (x^2) \sum_{n=0}^{\infty} p(2n)x^{2n} = \sum_{n=-\infty}^{\infty} 8n^2 - 2n
\end{equation}
(11)
and
\begin{equation}
\varphi^2 (x^2) \sum_{n=0}^{\infty} p(2n+1)x^{2n+1} = \sum_{n=-\infty}^{\infty} 8n^2 - 6n + 1
\end{equation}
(12)
that is
\begin{equation}
\varphi^2 (x^2) \sum_{n=0}^{\infty} p(2n+1)x^{2n} = \sum_{n=-\infty}^{\infty} 8n^2 - 6n
\end{equation}
(13)
To prove (6) we eliminate the \( \varphi^2 (x^2) \) term from (11) and (13) and equate coefficients of like powers of \( x \).

To prove (7) we rewrite (11) in the form:
\begin{equation}
\varphi(x^2) \sum_{n=0}^{\infty} p(2n)x^{2n} = \frac{1}{\varphi(x^2)} \sum_{k=-\infty}^{\infty} 8k^2 - 2k
\end{equation}
and using (1) and (2) we derive:
\begin{equation}
\sum_{k=-\infty}^{\infty} (-1)^k x^{3k^2 + k} \sum_{n=0}^{\infty} p(2n)x^{2n} = \sum_{n=0}^{\infty} p(n)x^{2n} \sum_{k=-\infty}^{\infty} 8k^2 - 2k
\end{equation}
(14)
now (7) follows from (14) on equating coefficients of \( x \). Similarly (8) follows from (13).

3. Further Identities For \( p(n) \)

Let \( N \) be a non-negative integer and let \( k, l, r, s \) be integers. Define:
\[ a_N = \sum_{k=-\infty}^{\infty} (-1)^k, \quad 48N + 5 = 4(6k + 1)^2 + (24l + 1)^2 \]
\[ b_N = \sum_{k=-\infty}^{\infty} (-1)^r, \quad 48N + 5 = 4(6r + 1)^2 + (24s + 1)^2 \]
\[ c_N = \sum_{48N + 29 = 4(6k + 1)^2 + (24l + 19)^2} (-1)^k, \]
\[ d_N = \sum_{48N + 29 = 4(6r + 1)^2 + (24s + 13)^2} (-1)^r. \]

Then the following holds:

**Theorem 2.**

\[ \sum_{\nu \geq 0, \ m \geq 0} (-1)\nu (2\nu + 1)p(4m) + \sum_{\nu \geq 0, \ m \geq 0} (-1)\nu (2\nu + 1)p(4m + 2) = a_N - b_N. \]

\[ \frac{1}{2} \nu (\nu + 1) + 2m = N \]

\[ \frac{1}{2} \nu (\nu + 1) + 2m = N - 1 \]  
(15)

\[ \sum_{\nu \geq 0, \ m \geq 0} (-1)\nu (2\nu + 1)p(4m + 1) + \sum_{\nu \geq 0, \ m \geq 0} (-1)\nu (2\nu + 1)p(4m + 3) = c_N - d_N. \]

\[ \frac{1}{2} \nu (\nu + 1) + 2m = N \]

\[ \frac{1}{2} \nu (\nu + 1) + 2m = N - 1 \]  
(16)

**Proof.** We use the formulas (1.5), (1.7) and (3.7) as given in [2]. Thus

\[ \sum_{0}^{\infty} p(4m)x^{2m} + \sum_{0}^{\infty} p(4m + 2)x^{2m + 1} = \frac{\varphi(x^2)}{\varphi^3(x)}\varphi(x^{24})A_1(x), \]

(17)

where

\[ \varphi(x^{24})A_1(x) = \sum_{-\infty}^{\infty} x^{(12l + 1)} - \sum_{-\infty}^{\infty} x^{(3s + 1)(4s + 1)} \]

(18)

Combining the above formulas we obtain

\[ \varphi^3(x) \left\{ \sum_{0}^{\infty} p(4m)x^{2m} + \sum_{0}^{\infty} p(4m + 2)x^{2m + 1} \right\} \]

\[ = \sum_{k, l = -\infty}^{\infty} (-1)^k x^{3k^2 + k + 12l^2 + l} - \sum_{r, s = -\infty}^{\infty} (-1)^r x^{3r^2 + 12s^2 + 7s + 1} \]

\[ = \sum_{N = 0}^{\infty} (a_N - b_N)x^N. \]

(19)

Now (15) follows on equating the coefficients of like powers of \( x \) in (19).
The proof of (16) actually follows the lines of the previous case, but we do not have a formula analogous to (18), but which we now develop. Using the formulas (1.6) and (1.8) in [2] we have:

$$\sum_{m=0}^{\infty} p(4m + 1)x^{2m} + \sum_{m=0}^{\infty} p(4m + 3)x^{2m + 1} = \frac{\varphi(x^2)}{\varphi^3(x)} \varphi(x^{24})A_3(x)$$  \hspace{1cm} (20)

where

$$A_3(x) = \prod_{m=1}^{\infty} (1 + x^{24m - 17})(1 + x^{24m - 7})$$

$$- x^2 \prod_{m=1}^{\infty} (1 + x^{24m - 23})(1 + x^{24m - 1})$$  \hspace{1cm} (21)

(See [2], page 348.) We need to have $\varphi(x^{24})A_3(x)$ expressed in the form of an infinite series. Let us put $y = x^{-5}$, $z = x^{12}$ in (5). Then

$$\varphi(x^{24}) \prod_{m=1}^{\infty} (1 + x^{24m - 17})(1 + x^{24m - 7}) = \sum_{k=-\infty}^{\infty} x^{12k^2 - 5k}$$

$$= \sum_{l=-\infty}^{\infty} x^{(l+1)(12l+7)}$$  \hspace{1cm} (22)

Similarly letting $y = x^{-11}$ and $z = x^{12}$ we get

$$x^2 \varphi(x^{24}) \prod_{m=1}^{\infty} (1 + x^{24m - 23})(1 + x^{24m - 1}) = \sum_{k=-\infty}^{\infty} x^{12k^2 - 11k + 2}$$

$$= \sum_{s=-\infty}^{\infty} x^{(4s+3)(3s+1)}$$  \hspace{1cm} (23)

Combining (20), (21), (22) and (23) we derive

$$\varphi^3(x) \left\{ \sum_{m=0}^{\infty} p(4m + 1)x^{2m} + \sum_{m=0}^{\infty} p(4m + 3)x^{2m + 1} \right\}$$

$$= \varphi(x^2) \left\{ \sum_{l=-\infty}^{\infty} x^{(l+1)(12l+7)} - \sum_{m=0}^{\infty} x^{(4s+3)(3s+1)} \right\}$$  \hspace{1cm} (24)

Now (16) follows on the lines of the proof of Theorem 2.
Corollary 1. If $48N + 5$ is not a sum of two squares then:

$$
\sum_{v \geq 0, \; m \geq 0} p(4m) + \sum_{v \geq 0, \; m \geq 0} p(4m + 3) \equiv 0 \pmod{4} \tag{25}
$$

$$
\frac{1}{2} v (v + 1) + 2m = N \quad \frac{1}{2} v (v + 1) + 2m = N - 1
$$

If $48N + 29$ is not a sum of two squares then:

$$
\sum_{v \geq 0, \; m \geq 0} p(4m + 1) + \sum_{v \geq 0, \; m \geq 0} p(4m + 3) \equiv 0 \pmod{4} \tag{26}
$$

$$
\frac{1}{2} v (v + 1) + 2m = N \quad \frac{1}{2} v (v + 1) + 2m = N - 1
$$

Proof. Since $\psi^3(x) = \sum_{n = 0}^{\infty} x^{(n + 1)/2} \pmod{4}$, then (25) and (26) follow immediately from (15) and (16) respectively.

Remark 1. We apply (6) to provide a new proof of Kolberg's result, that is $p(n)$ takes both even and odd values, each of them infinitely often.

Assume first that $p(n) = 0 \pmod{2}$ for all $n \geq t$, and let $2n = 8t^2 + 6t$. Then

$$
p(8t^2 + 6t + 1) + \ldots + p(12t) + p(0) \equiv 1 \pmod{2}
$$

contradiction.

If $p(n) = 1 \pmod{2}$ for all $n \geq t$, then we set $2n = 8t^2 + 2t$ and:

$$
p(8t^2 + 2t + 1) + \sum_{0 \leq k \leq -1} \{ p(8t^2 + 2t + 1 - 8k^2 + 2k) + p(8t^2 + 2t + 1 - 8k^2 - 2k) \} + \{ p(4t + 1) + p(1) \} \equiv 1 \pmod{2}
$$

since we have an odd number of odd terms. On the other hand:

$$
p(8t^2 + 2t) + \sum_{0 \leq k \leq -1} \{ p(8t^2 + 2t - 8k^2 + 6k) + p(8t^2 + 2t - 8k^2 - 6k) \} + p(8t) \equiv 0 \pmod{2}
$$

since now, there are an even number of odd terms.
Added in Proof: F. Garvan and D. Stanton (unpublished) have recently verified the Subbarao conjecture for $a = 3, 5$ and 10.

References


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