Two New Combinatorial Identities

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Separating the partitions of an integer into two classes according as the largest (smallest) part of the partition is repeated or not, we obtain two apparently new identities for $\varphi(a, x)$ and $1/\varphi(a, x)$, where $\varphi(a, x) = \prod_{n=1}^{\infty} (1 - ax^n)$.

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1. *The identities.* In this paper, we establish the following two apparently new identities by purely combinatorial arguments.

$$(1.1)\quad \prod_{n=1}^{\infty} (1 - ax^n) = 1 - \frac{ax}{1-x} + \frac{a^2}{1-x} \sum_{r=1}^{\infty} x^{2r+1}(1 - ax) \ldots (1 - ax^{r-1}) ;$$

$$(1.2)\quad \prod_{n=1}^{\infty} (1 - ax^n)^{-1} = 1 + \frac{ax}{1-x} + \frac{a^2}{1-x} \sum_{r=1}^{\infty} \frac{x^{2r}}{(1-ax) \ldots (1-ax^r)}$$

(For convergence purposes, we assume throughout that $|x| < 1$).

These identities are in contrast to the well known Eulerian expansions (Bellman 1961, p. 49):

$$(1.3)\quad \prod_{n=1}^{\infty} (1 - ax^n) = 1 + \sum_{r=1}^{\infty} \frac{(-1)^r a^r x^{r(r+1)/2}}{(1-x) \ldots (1-x^r)} ;$$

$$(1.4)\quad \prod_{n=1}^{\infty} (1 - ax^n)^{-1} = 1 + \sum_{r=1}^{\infty} \frac{a^r x^r}{(1-x) \ldots (1-x^r)}$$

We observe in passing, that (1.2) for $a = 1$ is the same as (1.4) for $a = X$.

2. *Proof of identity* (1.1). Throughout the paper, we use the notation:

$$\varphi(a, x) = \prod_{n=1}^{\infty} (1 - ax^n) ;$$

$$\varphi_r(a, x) = \prod_{n=1}^{r} (1 - ax^n), \ r > 0 ;$$

$$\varphi_0(a, x) = 1.$$

Let $S_0(n)$ denote the set of all the unrestricted partitions of n; $S_1(n)$ (respectively $S_2(n)$) the subset consisting of those partitions of n for which the smallest part is unity (respectively, greater than unity); and $S_3(n)$ (respectively, $S_4(n)$) the subset of partitions of n for which the smallest part is repeated (respectively, not repeated).

Associate with each partition of n a weight a^r, where r denotes the number of parts used in the partition (repeated parts being counted according to their multiplicities). Let $w_i(n) = w_i(a, n)$ denote the sum of all the weights associated with all the partitions of n that belong to $S_i(n)$, ($i = 0, 1, 2, 3, 4,$). Following the convention that empty sums have the value 0 and empty products the value unity, we have $w_2(1) = w_3(1) = 0$.

We observe the following obvious relations:

(2.1) $w_0(n) = w_1(n) + w_2(n)$.

(2.2) $w_0(n) = w_3(n) + w_4(n)$.

$$1 + \sum_{n=1}^{\infty} w_0(n)x^n = \frac{1}{\varphi(a, x)}$$

$$\sum_{n=1}^{\infty} w_1(n)x^n = \frac{ax}{\varphi(a, x)}. \quad (2.4)$$

Next, we shall show that

$$\sum_{n=2}^{\infty} w_3(n-1)x^n = \frac{a^2}{\varphi(a, x)} \sum_{r=1}^{\infty} x^{2r+1}\varphi_{r-1}(a, x). \quad (2.5)$$

For $n \geq 3$, consider all partitions of $n - 1$ in $S_3(n - 1)$ for each of which the smallest part is $r (r \geq 1)$. By the definition of $S_3(n - 1)$, his smallest part occurs at least twice in each of these partitions. Hence the sum of the weights of all such partitions of $n - 1$ is the coefficient of x^{n-1} in

$$\frac{ax^r \cdot ax^r}{(1 - ax^r)(1 - ax^{r+1}) \dots} = \frac{a^2x^{2r}\varphi_{r-1}(a, x)}{\varphi(a, x)}.$$

Letting $r = 1, 2, \dots$, and taking the sum of the weights in each case, we obtain (2.5).

In the next placc, (2.2) gives

$$\sum_{n=1}^{\infty} w_4(n-1)x^n = \sum_{n=2}^{\infty} w_0(n-1)x^n - \sum_{n=2}^{\infty} w_3(n-1)x^n,$$

so that, on using (2.3) and (2.5), we obtain

$$(2.6)\quad \sum_{n=2}^{\infty} w_4(n-1)x^n = \frac{x}{\varphi(a, x)} - x - \frac{a^2}{\varphi(a, x)} \sum_{r=1}^{\infty} x^{2r+1}\varphi_{r-1}(a, x) .$$

Finally, for any $n > 1$, we set up a one-to-one mapping F of $S_2(n)$ onto $S_4(n-1)$ as follows: given any member β of $S_2(n)$, we define its map $F(\beta)$ to be the partition of $n - 1$ obtained by reducing the smallest part of β by unity. Clearly, $F(\beta)$ belongs to $S_4(n-1)$ and the weights of β and $F(\beta)$ are the same, so that

$$w_2(n) = w_4(n-1), \quad (n = 2, 3, \ldots) .$$

Recalling that $w_2(1) = 0$ and using (2.6) we have

$$(2.7)\quad \sum_{n=1}^{\infty} w_2(n)x^n = \frac{x}{\varphi(a, x)} - x - \frac{a^2}{\varphi(a, x)} \sum_{r=1}^{\infty} \varphi_{r-1}(a, x)x^{2r+1} .$$

Hence (2.1), (2.3), (2.4) and (2.7) give

$$\frac{1}{\varphi(a, x)} = 1 + \frac{ax}{\varphi(a, x)} + \frac{x}{\varphi(a, x)} - x - \frac{a^2}{\varphi(a, x)} \sum_{r=1}^{\infty} \varphi_{r-1}(a, x)x^{2r+1} .$$

Multiplying both sides by $\varphi(a, x)$ this gives (1.1) after some simplification.

3. *Proof of identity* (1.2). We use similar arguments, taking this time the largest part in the partition. Let $S_5(n)$ (respectively $S_6(n)$) denote the set of those partitions of n in which the largest part is not repeated (respectively, repeated), and $w_5(n)$ and $w_6(n)$ the sum of the weights of the partitions belonging to $S_5(n)$ and $S_6(n)$ respectively.

For any $n > 1$, there is a one-to-one mapping G of $S_5(n)$ onto $S_0(n-1)$ defined as follows: if β belongs to $S_5(n)$, $G(\beta)$ is defined as the partition of $n - 1$ obtained by reducing the largest part in β by unity. Evidently, G preserves the weight of β: $w(\beta) = w(G(\beta))$. Noting that $w(1) = a$, we thus obtain

$$(3.1)\quad \sum_{n=1}^{\infty} w_5(n)x^n = ax + \sum_{n=2}^{\infty} w_0(n-1)x^n = ax + \frac{x}{\varphi(a, x)} - x .$$

Also, arguing as for the proof of (2.5) we have (with $w_6(1) = 0$)

$$\sum_{n=1}^{\infty} w_6(n)x^n = \sum_{n=1}^{\infty} \frac{a^2x^{2r}}{(1-ax)\ldots(1-ax^r)} .$$

Since $w_0(n) = w_5(n) + w_6(n)$, $(n = 1,2,\ldots)$, we have finally

$$\frac{1}{\varphi(a, x)} = 1 + \sum_{n=1}^{\infty} w(n)x^n = 1 + \sum_{n=1}^{\infty} w_5(n)x^n + \sum_{n=1}^{\infty} w_6(n)x^n$$

$$= 1 + ax - x + \frac{x}{\varphi(a, x)} + a^2 \sum_{n=1}^{\infty} \frac{x^{2r}}{\varphi_r(a, x)},$$

from which (1.2) follows.

REFERENCES

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