DISTRIBUTION
OF 2–ADDITIVE FUNCTIONS
UNDER SOME CONDITIONS

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Abstract. Distribution of 2-additive functions under the condition
α(n) = k is investigated, where α(n) is the sum of digits in the binary
expansion of n.

1. Introduction and formulation of the theorems

Let ε_j(n) be the j' th digit in the binary expansion of n,

(1.1) \[ n = \sum_{j=0}^{\infty} \varepsilon_j(n) \cdot 2^j, \quad \varepsilon_j(n) \in \{0, 1\}. \]

Let \( A_2 \) be the class of 2-additive and \( M_2 \) be the class of 2-multiplicative
functions.
A function \( f : \mathbb{N}_0(= \mathbb{N} \cup \{0\}) \rightarrow \mathbb{R} \) belongs to \( A_2 \), if

\[ f(0) = 0, \text{ and } f(n) := \sum_{j=0}^{\infty} \varepsilon_j(n) f(2^j), \]

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and \( g : \mathbb{N}_0 \to \mathbb{C} \) belongs to \( \mathcal{M}_2 \), if

\[
g(0) = 1, \quad \text{and} \quad g(n) := \prod_{j=0}^{\infty} g(e_j(n) \cdot 2^j).
\]

Let \( \overline{\mathcal{M}}_2 \) be the set of those \( g \in \mathcal{M}_2 \) for which additionally \( |g(n)| = 1 \) \( (n \in \mathbb{N}_0) \) holds.

Let \( \alpha(n) = \sum_{j=0}^{\infty} e_j(n) \) be the so called "sum of digits" function.

Let \( \mathcal{E}_{N,k} = \{ n < 2^N \mid \alpha(n) = k \} \), and

\[
\eta = \eta_{N,k} = \frac{k}{N}.
\]

Here we continue our work [1].

**Theorem 1.** Let \( g \in \overline{\mathcal{M}}_2 \) be such a function for which

\[
\sum_{j=0}^{\infty} (1 - g(2^j))
\]

is convergent. Let

\[
M_\eta := \prod_{j=0}^{\infty} ((1 - \eta) + g(2^j)\eta).
\]

Let \( \delta > 0 \) be a constant. Then

\[
\max_{\delta \leq \frac{k}{N} \leq 1 - \delta} \left| \frac{1}{N} \sum_{n \in \mathcal{E}_{N,k}} g(n) - M_{\eta_{N,k}} \right| \to 0 \quad (N \to \infty).
\]

**Theorem 2.** Let \( f \in \mathcal{A}_2 \) such that \( \sum f(2^j) \), \( \sum f^2(2^j) \) are convergent. Let \( \varphi_\eta(z) \) be the characteristic function of \( \Theta = \xi_0 + \xi_1 + \ldots \), where \( \xi_0, \xi_1, \ldots \) are independent random variables,

\[
P(\xi_0 = 0) = 1 - \eta, \quad P(\xi_0 = f(2^j)) = \eta.
\]
Thus
\[ \varphi_\eta(\tau) = \prod_{j=0}^{\infty} \left( (1 - \eta) + \eta \cdot e^{i\tau f(2^j)} \right). \]

Let \( F_\eta(y) \) be the distribution function of \( \Theta \).

Then
\[ \max_{\delta \leq \tau \leq 1 - \delta} \sup_{y \in \mathbb{R}} \left| \frac{1}{N \choose k} \# \{ n \in \mathcal{E}_{N,k}, \ f(n) < y \} - F_\eta(y) \right| \rightarrow 0 \quad (N \to \infty). \]

Here \( \delta > 0 \) is an arbitrary small constant.

**Theorem 3.** Let \( f \in A_2, \ f(2^j) = O(1) \). Let \( A_N = \sum_{j=0}^{N-1} f(2^j), \ m_N(\eta) := \eta A_N, \)
\[ \sigma_N^2(\eta) = (1 - \eta)\eta \sum_{j=0}^{N-1} \left( f(2^j) - \frac{A_N}{N} \right)^2 \]
\( \eta \in [\delta, \ (1 - \delta)], \ \delta > 0 \) be a constant.

Assume that \( \sigma_N^2 \left( \frac{1}{2} \right) \to \infty \quad (N \to \infty). \) Then
\[ \lim_{N \to \infty} \sup_{\delta \in [\delta, \ 1 - \delta]} \sup_{y \in \mathbb{R}} \left| \frac{1}{N \choose k} \# \left\{ n \in \mathcal{E}_{N,k}, \ \left| \frac{f(n) - m_N(\frac{k}{N})}{\sigma_N(\frac{k}{N})} < y \right\} - \Phi(y) \right| = 0. \]

The proof of this last theorem is very similar to the proof of Theorem 3 in [1], so we omit it.

**2. Proof of Theorem 1 and 2**

It is enough to prove Theorem 1. Theorem 2 follows hence, if we consider \( g_\tau(n) = e^{i\tau f(n)} \) and apply Theorem 1.

The proof is almost the same as that of Theorem 2 in [1].
Let $M$ be a large fixed integer, $\arg g(2^j) = h(2^j), \ h(2^j) \in [-\pi, \pi]$. From (1.2) we obtain that
\[
\sum |1 - g(2^j)|^2 < \sum h^2(2^j) < \infty,
\]
and that $\sum h(2^j)$ is convergent. Thus $g(2^j) \to 1 \ (j \to \infty)$. Let $h$ be defined on $\mathbb{N}_0$ as a 2-additive function. Then $g(n) = e^{ih(n)}$.

Let
\[
g_M(n) = \prod_{j=0}^{M-1} g(\varepsilon_j(n) \cdot 2^j), \quad h_M(n) = \sum_{j=0}^{M-1} h(\varepsilon_j(n) \cdot 2^j),
\]

\[
h_M^*(n) = \sum_{j=M}^{N-1} h(\varepsilon_j(n) \cdot 2^j).
\]

We have
\[
\frac{1}{\binom{N}{k}} \sum_{n \in \mathcal{E}_{N,k}} h_M^*(n) = \sum_{j=M}^{N-1} h(2^j) \cdot \binom{N-1}{k-1} \cdot \binom{N}{k},
\]

\[
\frac{1}{\binom{N}{k}} \sum_{n \in \mathcal{E}_{N,k}} h_M^2(n) = \sum_{j=M}^{N-1} h^2(2^j) \cdot \binom{N-1}{k-1} \cdot \binom{N}{k} +
\]

\[
+ \sum_{M \leq j_1 < j_2 \leq N-1} \binom{N-1}{k-2} \cdot \binom{N}{k} \cdot h(2^{j_1}) h(2^{j_2}).
\]

Furthermore
\[
\frac{(N-1)}{\binom{N}{k}} = \frac{k}{N} = \eta, \quad \frac{(N-2)}{\binom{N}{k}} = \frac{k(k-1)}{N(N-1)} = \eta^2 \left(1 + O\left(\frac{1}{N}\right)\right).
\]
Hence we obtain that
\[
\frac{1}{N} \sum_{n \in \mathcal{E}_{N,k}} \left( h_M(n) - \eta \sum_{j=M}^{N-1} h(2^j) \right)^2 \ll \eta \sum_{j=M}^{N-1} h^2(2^j) + \frac{1}{N} \sum_{M \leq i, j \leq N-1} |h(2^i)| \cdot |h(2^j)|.
\]

The right hand side tends to zero as \( M \to \infty \). It implies that
\[
\limsup_{N \to \infty} \max_{\frac{1}{2} \leq \delta \leq 1-\delta} \left| \frac{1}{N} \sum_{n \in \mathcal{E}_{N,k}} (g(n) - g_M(n)) \right| = \Delta(M) \to 0 \quad \text{as } M \to \infty.
\]

To estimate \( \frac{1}{N} \sum_{n \leq 2N} g_M(n) \), we write each \( n \in \mathcal{E}_{N,k} \) as \( n = t + q^M m \).

For a fixed \( t, n \in \mathcal{E}_{N,k} \) if and only if \( m \in \mathcal{E}_{N-M, k-\alpha(t)} \), thus
\[
\frac{1}{N} \sum_{n \leq 2N} g_M(n) = \sum_{t=0}^{2^{M-1}} g(t) \cdot \frac{N-M}{(N/k)^{\alpha(t)}} =
\]
\[
= \sum_{t=0}^{2^{M-1}} g(t) \left( \frac{\eta}{1-\eta} \right)^{\alpha(t)} (1-\eta)^M (1+o_N(1)) =
\]
\[
= (1+o_N(1))(1-\eta)^M \sum_{t=0}^{2^{M-1}} g(t) \left( \frac{\eta}{1-\eta} \right)^{\alpha(t)} =
\]
\[
= (1+o_N(1))(1-\eta)^M \prod_{j=0}^{M-1} \left( 1 + g(2^j) \frac{\eta}{1-\eta} \right) =
\]
\[
= (1+o_N(1)) \prod_{j=0}^{M-1} \left( (1-\eta) + g(2^j)\eta \right).
\]
The relation is uniform as \( \frac{k}{N} \in [\delta, 1-\delta] \). Hence the theorem is immediate.
3. Final remarks

We can prove the following assertions.

**Theorem 4.** Let \( g \in \mathcal{M}_2, \delta > 0 \) and assume that there is a sequence \( k_N = k \) such that

\[
\frac{1}{\binom{N}{k}} \sum_{n \in \mathcal{E}_{N,k}} g(n) - M_{\eta_N,k} \to 0 \quad \text{as} \quad N \to \infty, \quad k = k_N.
\]

Then (1.2) is convergent.

**Theorem 5.** Let \( f \in \mathcal{A}_2, \delta > 0 \), and assume that for a suitable sequence \( k = k_N \) such that \( \eta \in (\delta, 1 - \delta) \) we have

\[
\sup_{y \in \mathbb{R}} \left| \frac{1}{\binom{N}{k}} \# \{ n \in \mathcal{E}_{N,k}, f(n) < y \} - F_{\eta_N,k}(y) \right| \to 0
\]

as \( N \to \infty, \ k = k_N \). Then the series \( \sum f(2^j), \sum f^2(2^j) \) are convergent.

We shall prove these assertions in more general form in a subsequent paper.

**Reference**


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