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THE CHARACTERIZATION OF n^{ir} AS A MULTIPLICATIVE FUNCTION

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Abstract. the following assertion is proved. If f is a completely multiplicative function taking values of modulus 1, such that the set $\mathcal{E}$ of limit points of $f(n+1)\overline{f}(n)$ has $k \leqq 3$ distinct values, then $f(n) = n^{ir}F(n)$ with some $r \in \mathbf{R}$, and $F^k(n) = 1$ for every n.

1. Let $\mathcal{M}$ be the class of those complex valued completely multiplicative functions f for which $|f(n)| = 1$ $(n = 1, 2, \dots)$ holds true. For some $f \in \overline{\mathcal{M}}$ let $\mathcal{E}_f$ denote the set of all those $z \in \mathbf{C}$ for which there exists an infinite sequence $n_\nu \to \infty$ such that $f(n_\nu + 1)\overline{f}(n_\nu) \to z$ $(\nu \to \infty)$ holds true. Let S_k be the set of the k-th roots of unity, i.e. $S_k = \{\omega | \omega^k = 1\}$. We shall formulate the following conjectures.

CONJECTURE 1. If $f \in \overline{\mathcal{M}}$, $\mathcal{E}_f$ is a finite set $\#(\mathcal{E}_f) = k$, then $\mathcal{E}_f = S_k$ and $f(n) = n^{i\tau}F(n)$ with some $\tau \in \mathbf{R}$, where $F^k(n) = 1$ $(n \in \mathbf{N})$.

CONJECTURE 2. If $f \in \mathcal{M}$ and $\mathcal{E}_f$ is infinite, then $\mathcal{E}_f = \{z | |z| = 1\}$.

REMARKS. 1. Conjecture 2 is not true for the whole set of the multiplicative functions f with $|f(n)| = 1$ $(n \in \mathbf{N})$. E.g. let $f(n) = 1$ for odd integers, and $f(2^k) = e^{2\pi i/k}$ $(k = 1, 2, \dots)$.

2. Conjecture 1 for $\mathcal{E}_f = \{1\}$ has been proposed by I. Kátai, and solved by E. Wirsing in 1984. His proof is published in [1].

3. Conjecture 1 contains as a special case the following assertion, which we formulate now as

CONJECTURE 3. Let $F \in \overline{\mathcal{M}}$, $F(\mathcal{N}) \subseteqq S_k$. Assume that there is an n_0 for which $F(n_0)$ is a primitive k-th root of unity. Then $\{F(n+1)\overline{F}(n) | n \in \mathbf{N}\} = S_k$.

THEOREM. *Conjecture* 1 *is true for* $k = 1, 2, 3$.

2. PROOF. Let $f \in \overline{\mathcal{M}}$ and $\mathcal{E}_f = \{\alpha_1, \dots, \alpha_k\}$ be a finite set, $\delta = \min_{i \neq j} |\alpha_i - \alpha_j|$. For every large n $(n > n_0$, say), there is exactly one $\alpha \in \mathcal{E}_f$ for

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which $|f(n+1)\overline{f}(n)-\alpha| < \delta$. Let $c(n) := \alpha$, i.e. $c(n)$ is that element of $\mathcal{E}_f$ which is closest to $f(n+1)\overline{f}(n)$. Since

$$\frac{f(n+1)}{f(n)} = \frac{f\big(d(n+1)\big)}{f(dn)} = \prod_{j=0}^{d-1} \frac{f(dn+j+1)}{f(dn+j)},$$

therefore for each $n > n_1(d)$ we have

$$c(n) = \prod_{j=0}^{d-1} c(dn+j). \tag{2.1}$$

Furthermore $f(n^2-1)\overline{f}(n^2) = f(n+1)\overline{f}(n)f(n-1)\overline{f}(n)$, whence

$$c(n-1) = c(n)c(n^2-1) \quad \text{if} \quad n > n_2 \tag{2.2}$$

follows, where n_2 is a suitable constant.

The case $k=1$. Then $\mathcal{E}_f = \{\alpha_1\}$. From (2.2) we get that $\alpha_1^2 = \alpha_1$, i.e. $\alpha_1 = 1$, and we can refer to Wirsing's theorem.

The case $k=2$. Let $\mathcal{E}_f = \{\alpha_1, \alpha_2\}$. Since the sequence $\{c(n) \mid n = 1, 2, \dots\}$ takes both of the values α_1, α_2 infinitely many times, therefore both of (α_1, α_2), (α_2, α_1) occur in $\big(c(n-1), c(n)\big)$ for infinitely many n. Consequently, from (2.2) we get that $\alpha_1\overline{\alpha}_2 \in \mathcal{E}_f$, $\alpha_2\overline{\alpha}_1 \in \mathcal{E}_f$.

Assume first that $\alpha_1\overline{\alpha}_2 = \alpha_2\overline{\alpha}_1$, i.e. that $\alpha_1^2 = \alpha_2^2$, whence $\alpha_2 = -\alpha_1$. Since $\mathcal{E}_{f^2} = \{\alpha_1^2, \alpha_2^2\} = \{\alpha_1^2\}$, from the already proved part of the theorem (case $k=1$) we get that $\alpha_1^2 = 1$, i.e. $\mathcal{E}_f = \{1, -1\}$. Since $f^2(n+1)\overline{f}^2(n) \to 1$, therefore, from Wirsing's theorem $f^2(n) = n^{ir}$, consequently $\big(f(n)n^{-ir/2}\big)^2 = 1$. Thus the theorem is true for $F(n) = f(n)n^{-ir/2}$.

Assume now that $\alpha_1\overline{\alpha}_2 \neq \alpha_2\overline{\alpha}_1$. Then either $\alpha_1\overline{\alpha}_2 = \alpha_1$ and $\alpha_2\overline{\alpha}_1 = \alpha_2$, or $\alpha_1\overline{\alpha}_2 = \alpha_2$ and $\alpha_2\overline{\alpha}_1 = \alpha_1$. In the first case $\overline{\alpha}_2 = \overline{\alpha}_1 = 1$, which is impossible. In the second case $\alpha_1 = \alpha_2^2$, $\alpha_2 = \alpha_1^2$, whence $\alpha_1 = \alpha_1^4$, $1 = \alpha_1^3$ follows. If $\alpha_1 = 1$, then $\alpha_2 = 1$, and this is impossible. So we may assume that $1 \notin \mathcal{E}_f$. But then $c(n-1) = c(n)$ cannot hold for $n > n_2$ (see (2.2)). We have $\mathcal{E}_f = \{\omega, \overline{\omega}\}$, where ω is a primitive third root of unity. From (2.1) we obtain that

$$c(n) = c(2n)c(2n+1) \quad \text{if} \quad n > n_1(2),$$

and this with $c(2m) \neq c(2n+1)$ implies that $c(n) = 1$. This is a contradiction.

We proved the theorem for $k=2$.

The case $k=3$. Let $\mathcal{E}_f=\{\alpha_1,\alpha_2,\alpha_3\}$. Let $G(\mathcal{E}_f)$ denote the directed graph with nodes $\alpha_1,\alpha_2,\alpha_3$ which is obtained by drawing arrows from α_i to α_j for all those α_i,α_j for which $c(n-1)=\alpha_i$, $c(n)=\alpha_j$ occurs for at least one $n>n_2$. From (2.2) we obtain that $\alpha_i\to\alpha_j$ implies that $\alpha_i\overline{\alpha}_j\in\mathcal{E}_f$. Moreover, if there is an integer $n>n_1(2)$ such that $c(2n)=\alpha_u$, $c(2n+1)=\alpha_v$, then $\alpha_u\alpha_v\in\mathcal{E}_f$ as well.

Case A. Assume that $1\notin\mathcal{E}_f$. Then there is no loop in $G(\mathcal{E}_f)$, since $\alpha_u\to\alpha_u$ would imply that $1=\alpha_u\overline{\alpha}_u\in\mathcal{E}_f$.

Since $c(n)=c(2n)c(2n+1)$, and $c(n)=\alpha_i$ holds for infinitely many n for every $\alpha_i\in\mathcal{E}_f$, there exists α_u and α_v, $u=v$ is not excluded, such that $\alpha_i=\alpha_u\alpha_v$. Since $1\notin\mathcal{E}_f$, therefore $\alpha_u\neq\alpha_i$, $\alpha_v\neq\alpha_i$. Since $c(n)\neq c(n+1)$ holds for $n>n_2$, therefore $\alpha_1=\alpha_2\alpha_3$, $\alpha_2=\alpha_1\alpha_3$, $\alpha_3=\alpha_1\alpha_2$. Multiplying these inequalities we deduce that $\alpha_1\alpha_2\alpha_3=(\alpha_1\alpha_2\alpha_3)^2$, $\alpha_1\alpha_2\alpha_3=1$, whence $\alpha_j^2=1$ $(j=1,2,3)$ holds. This is impossible.

Case B. $\mathcal{E}_f=\{1,\alpha,\beta\}$.

Case B1. Let $\alpha=-1$. Then $\mathcal{E}=\{1,-1,\beta\}$, and there exist infinitely many n for which $c(n-1)\in\{1,-1\}$ and $c(n)=\beta$. Consequently $c(n-1)\overline{c}(n)=\pm\overline{\beta}$, i.e. either $-\overline{\beta}\in\mathcal{E}_f$, or $\overline{\beta}\in\mathcal{E}_f$. This is impossible, since $\overline{\beta}\neq\beta$, $1,-1$.

Case B2. Let $-1\notin\mathcal{E}_f$. Then $\beta=\overline{\alpha}$. Indeed, if n is such a large integer for which $c(n-1)=1$, $c(n)\neq1$, then by (2.2), $\overline{c(n)}\in\mathcal{E}_f$.

If $c(n-1)=\overline{c(n)}=\alpha$ or $\overline{\alpha}$ occurs for some $n>n_2$, then α^2 (or $\overline{\alpha}^2$) $\in\mathcal{E}_f$, and so $\alpha^2=\overline{\alpha}$ (or $\overline{\alpha}^2=\alpha$), whence $\mathcal{E}_f=\{1,\omega,\overline{\omega}\}$, ω is a primitive third root of unity. In this case $\mathcal{E}_{f^3}=\{1\}$, and by Wirsing's theorem $f^3(n)=n^{ir}$.

If there is such an $n>n_1(2)$ for which $c(2n)=c(2n+1)\neq1$, then we are ready as well, since $c(n)=c(2n)c(2n+1)$, which implies that α_2 (or $\overline{\alpha}^2$) belongs to $\mathcal{E}_f$, and as above, we conclude that $\mathcal{E}_f=\{1,\alpha,\overline{\alpha}\}$.

Hence we may assume that at least one of the elements $c(2n)$, $c(2n+1)$ is 1, if $n>n_1(2)$.

We can formulate now the following consequences:

(1) Let $c(n)=\alpha$ (or $\overline{\alpha}$), $n>2n_1(2)$, $n_1=[\frac{n}{2}]$. Then $c(n_1)=c(2n_1)c(2n_1+1)$, consequently $c(n_1)=\alpha$ (or $\overline{\alpha}$, respectively).

(2) If $c(n)=\alpha$ (or $\overline{\alpha}$), $n>n_1(2)$, then exactly one of $c(2n)$, $c(2n+1)$ is α (resp. $\overline{\alpha}$), the other equals 1.

(3) Let N be a large number for which $c(N)=\alpha$ (or $=\overline{\alpha}$). Let $N=\mathcal{E}_0 2^M+\mathcal{E}_1 2^{M-1}+\ldots+\mathcal{E}_M$ be the binary expansion of N, $N=\mathcal{E}_M+2N_1$, $N_1=\mathcal{E}_{M-1}+2N_2,\ldots$. Then there is a constant $K=2^T$ depending on f such that $c(N_j)=\alpha$ (resp. $\overline{\alpha}$) for all those j for which $N_j>K$. Observe that $N_j\geqq\frac{1}{2}N_{j-1}$. Since $c(n)=\alpha$ occurs for infinitely many n, therefore for every large t the interval $[2^t,2^{t+1})$ contains an integer N such that $c(N)=\alpha$.

(4) Let $N^{(1)} < N^{(2)}$, $N^{(j)} \in [2^t, 2^{t+1}]$ be such integers for which $c\left(N^{(1)}\right) = c\left(N^{(2)}\right) = \alpha$ (or $\overline{\alpha}$). Then $N^{(2)} - N^{(1)} > \frac{2^t}{2^T}$.

We consider only the case $c\left(N^{(1)}\right) = \alpha$. To prove this, define $N_l^{(j)} = \left[\frac{N^{(G)}}{2^l}\right]$ $(l = 0, 1, 2, \ldots)$. We have $c\left(N_l^{(j)}\right) = \alpha$, while $N_l^{(j)} > K$. Let l be the smallest integer for which $N_l^{(1)} = N_l^{(2)}$. Then $N_{l-1}^{(2)} - N_{l-1}^{(1)} = 1$, $c\left(N_{l-1}^{(1)}\right) = c\left(N_{l-1}^{(1)} + 1\right) = \alpha$. From (3) we obtain that $N_{l-1}^{(1)} \leqq K$. Since $N_{l-1}^{(1)} \in \left[\frac{2^t}{2^l - 1}, \frac{2^{t+1}}{2^{l-1}}\right]$, it follows that $N^{(2)} - N^{(1)} > 2^l\left(N_{l-1}^{(2)} - N_{l-1}^{(1)}\right) \geqq 2^l \geqq \frac{2^t}{2^T}$.

(5) One can prove similarly, that, if $N^{(1)} < N^{(2)}$, $c\left(N^{(1)}\right) = \xi$, $c\left(N^{(2)}\right) = \overline{\xi}$, $\xi \in \{\alpha, \overline{\alpha}\}$, then

$$N^{(2)} - N^{(1)} > \frac{N^{(1)}}{2^T}.$$

(6) From (4) and (5) we obtain immediately: for each fixed integer d there is a constant $m(d)$ such that for both values of $\xi = \alpha, \overline{\alpha}$,

– if $n > m(d)$ and $c(n) = \xi$, then $c\left(\left[\frac{n}{d}\right]\right) = \xi$,

– if $N > m(d)$ and $c(n) = \xi$ then one of $c(dn + j)$ $(j = 0, \ldots, d-1)$ is ξ.

To finish the proof we deduce that $c(2^u) = \alpha = \overline{\alpha}$ holds for every large u. This is impossible, since $\alpha \neq \pm 1$.

Let u be fixed, N be an integer such that $c(N) = \alpha$, and $N > \max_{2^u \leqq d < 2^{u+1}} n_1(d)$. Let N_s be defined by $N = N_s 2^s + l_s$ $(0 \leqq l_s < 2^s)$, where $2^u \leqq \overline{N}_s < 2^{u+1}$. Then, by (2.1), and from (6) we obtain that $c\left(\left[\frac{N}{N_s}\right]\right) = \alpha$. Since $\frac{N}{N_s} = 2^s + \frac{l_s}{N_s}$, $\left[\frac{N}{N_s}\right] = 2^s + f$, $0 \leqq f < 2^{s-u}$, therefore

$$\alpha = c\left(\left[\frac{1}{2^{s-u}}\left[\frac{N}{N_s}\right]\right]\right) = c(2^u).$$

Starting from such an N for which $c(N) = \overline{\alpha}$, we deduce that $\overline{\alpha} = c(2^u)$. The proof is complete.

Reference

[1] E. Wirsing, Tang Yuansheng and Shao Pintshung, On a conjecture of Kátai for additive functions, *J. Number Theory*, **56** (1996), 391–395.

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