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THE IDENTICAL EQUATION IN ψ-PRODUCTS

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ABSTRACT. In Bull. Amer. Math. Soc. **36** (1930), 762–772, R. Vaidyanathaswamy established a remarkable identity valid for any multiplicative arithmetic function and involving Dirichlet convolution. D. H. Lehmer (Trans. Amer. Math. Soc. **33** (1931), 945–952) introduced a very general class of arithmetical convolutions, called ψ-products, which include the well-known Dirichlet products, Eckford Cohen's unitary convolutions, and in fact Narkiewicz's so-called regular A-convolutions. In this paper, we establish an identical equation valid for multiplicative arithmetic functions and Lehmer's ψ-convolutions which yields, as special cases, all known identical equations valid for the Dirichlet and unitary convolutions, besides establishing identical equations for several new convolutions.

1. INTRODUCTION

An arithmetic function is a complex-valued function whose domain is the set of positive integers $\mathbb{Z}^+$. The set of all arithmetic functions will be denoted by F. If $f \in F$, then as usual, f is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$, for all positive integers m and n with $(m, n) = 1$; here the symbol (a, b) stands for the greatest common divisor of a and b.

In 1930, R. Vaidyanathaswamy (see [11] and [12, Section VI]) established the following remarkable identity valid for any multiplicative function and known as the identical equation for multiplicative functions: If f is any multiplicative function, then for any positive integers m and n, we have

$$f(mn) = \sum_{\substack{a|m \\ b|n}} f(m/a)f(n/b)f^{-1}(ab)G(a,b), \tag{1.1}$$

where f^{-1} is the inverse of f with respect to the familiar Dirichlet convolution i.e.,

$$\sum_{d|m} f(d)f^{-1}(m/d) = e(m),$$

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for all positive integers m, where

$$e(m) = \begin{cases} 1, & \text{if } m = 1, \\ 0, & \text{if } m > 1, \end{cases} \tag{1.2}$$

and

$$G(a,b) = \begin{cases} (-1)^{\omega(a)}, & \text{if } \gamma(a) = \gamma(b), \\ 0, & \text{otherwise;} \end{cases} \tag{1.3}$$

$\omega(a)$ being the number of distinct prime factors of a, $\gamma(a)$ the product of distinct prime factors of a with $\omega(1) = 0$ and $\gamma(1) = 1$.

The identical equation (1.1) attracted the attention of many mathematicians. A. A. Gioia [2] and M. Sugunamma (cf. [10, page 30]) offered different proofs of (1.1) while M. V. Subbarao and A. A. Gioia [9] and P. J. McCarthy [6] generalized (1.1), in different directions (see also K. Krishna [4]).

A divisor d of m is said to be a unitary divisor [1] if $(d, m/d) = 1$ and in such a case we write $d\|m$.

It has been observed by M. V. Subbarao and A. A. Gioia [9] that the unitary analogue of (1.1) is true i.e., whenever m and n are relatively prime and f is multiplicative we have

$$f(mn) = \sum_{\substack{a\|m \\ b\|m}} f(m/a)f(n/b)(f^*)^{-1}(ab)G(a,b), \tag{1.4}$$

where $(f^*)^{-1}$ is the inverse of f with respect to the unitary convolution [1]; that is,

$$\sum_{d\|m} f(d)(f^*)^{-1}(m/d) = e(m).$$

In fact, M. V. Subbarao and A. A. Gioia noted that (cf. [9, p. 70]) the identity in (1.4) reduces to a triviality in the sense that the right-hand side of (1.4) can be evaluated without much difficulty since $(f^*)^{-1}(m) = (-1)^{\omega(m)}f(m)$. They also established a non-trivial identity (cf. [9, Theorem 2]) in the case of unitary products.

As a generalization of the Dirichlet and unitary convolutions, W. Narkiewicz [7] introduced the concept of a regular A-convolution. It is interesting to note that the A-analogue of (1.1) is also true which has in fact been established by P. Haukkanen (cf. [3, Theorem 1.4.8], $G = \mathbb{Z}^+$) in a slightly more general setting. However, we mention here only the A-analogue of (1.1): If f is a multiplicative function, then we have for $m \in A(mn)$,

$$f(mn) = \sum_{\substack{a\in A(m) \\ b\in A(n)}} f(m/a)f(n/b)f_A^{-1}(ab)G(a,b), \tag{1.4'}$$

where f_A^{-1} is the inverse of f with respect to the regular A-convolution, so that

$$\sum_{d\in A(m)} f(d)f_A^{-1}(m/d) = e(m).$$

Let T be a non-empty subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$ and let $\psi\colon T \to \mathbb{Z}^+$ be a mapping satisfying the following conditions:

(1.5) For each $n \in \mathbb{Z}^+, \psi(x,y) = n$ has a finite number of solutions.

(1.6) If $(x,y) \in T$, then $(y,x) \in T$ and we have $\psi(x,y) = \psi(y,x)$.

(1.7) The statement "$(x,y) \in T, (\psi(x,y),z) \in T$" and "$(y,z) \in T, (x,\psi(y,z)) \in T$" are equivalent; if one of these conditions holds, we have

$$\psi(\psi(x,y),z) = \psi(x,\psi(y,z)).$$

(1.8) For each $k \in \mathbb{Z}^+, (1,k) \in T$ and $\psi(1,k) = k$.

If $f, g \in F$, then the ψ-product of f and g denoted by $f\psi g$ is defined by

$$(f\psi g)(n) = \sum_{\psi(x,y)=n} f(x)g(y), \tag{1.9}$$

for each $n \in \mathbb{Z}^+$.

The binary operation ψ in (1.9) is due to D. H. Lehmer [5]. It is not difficult to show that (see [5]) the triple $(F, +, \psi)$ is a commutative ring with unity e where e is as given in (1.2) and '+' denotes the usual pointwise addition.

Let $\psi(x,y) = xy$ for all $(x,y) \in T$. If $T = \mathbb{Z}^+ \times \mathbb{Z}^+$, then from (1.9) it is clear that ψ reduces to the Dirichlet convolution. If $T = \{(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \colon a$ and b are relatively prime$\}$, then ψ reduces to the unitary convolution [1]. More generally, if A is Narkiewicz's regular convolution [7] and $T = \bigcup_{n=1}^{\infty}\{(x, n/x) \colon x \in A(n)\}$, then ψ reduces to the A-convolution. Examples show that the binary operation ψ of D. H. Lehmer [5] is more general than that of Narkiewicz's A-convolution.

The object of the present paper is to obtain a 'ψ-analogue' of the identical equation (1.1).

When $T = \mathbb{Z}^+ \times \mathbb{Z}^+$ and $\psi(x,y) = xy$ on T, the result in (1.1) can be restated as follows: If f is multiplicative, then for any pair $(m,n) \in T$, we have

$$f(\psi(m,n)) = \sum_{\substack{\psi(a,x)=m \\ \psi(b,y)=n}} f(x)f(y)f^{-1}(\psi(a,b))G(a,b). \tag{1.10}$$

It is clear that we obtain the results in (1.4) and (1.4′), by taking $T = \{(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \colon a$ and b are relatively prime$\}$, $T = \bigcup_{n=1}^{\infty}\{(x, n/x) \colon x \in A(n)\}$, where A is a Narkiewicz convolution and $\psi(x,y) = xy$ on T in (1.10) successively.

In the case of Dirichlet, unitary or more generally that of a regular convolution, whenever f is multiplicative, f^{-1} exists and is also multiplicative (cf. [7]). When we are aiming at (1.10), of course, we need a ψ-function in which such a property is there. Moreover, the Dirichlet or unitary or in general a regular convolution is multiplicativity preserving in the sense that whenever f and g are multiplicative, then the corresponding product in each of these convolutions is also multiplicative. For an efficient evaluation of the right-hand side of (1.10), we are forced to consider the binary operation ψ which is multiplicativity preserving, that is, we need a ψ in which $f\psi g$ is multiplicative whenever f and g are so. With an additional restriction on ψ (apart from the conditions (1.5)–(1.8)), namely, $\psi(x,y) \geq \max\{x,y\}$, for all $(x,y) \in T$, in [8], we observed that (see §2, Lemmas 2.1 and 2.2) the characterization of multiplicativity-preserving ψ-functions is possible. Unfortunately, even when ψ is multiplicativity preserving and in which f^{-1} is multiplicative whenever f is so, the identity in (1.10) may not hold (see Remark 3.3). This leads us to impose further restrictions on ψ (see section 3). However, even with these restrictions, we can obtain examples of ψ-functions other than the familiar $\psi(x,y) = xy$ (see Remark 3.2) for which an identical equation holds.

Section 2 deals with the preliminaries. Section 3 is devoted to the main result of this paper.

2. Preliminaries

First we have

Lemma 2.1 (cf. [8], Theorem 3.1). *Let $\psi\colon T \to \mathbb{Z}^+$ be a mapping satisfying* (1.5)–(1.8). *Also, let $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$. Suppose that the binary operation ψ in* (1.9) *preserves multiplicativity, that is, whenever $f, g \in F$ and are multiplicative then so is $f\psi g$. If $x = \prod_{i=1}^r p_i^{\alpha_i}$ and $y = \prod_{i=1}^r p_i^{\beta_i}$, where $p_1, p_2, \ldots, p_r$ are distinct primes, and α_i and β_i are non-negative integers, we have*

(a) *$(x,y) \in T$ if and only if $(p_i^{\alpha_i}, p_i^{\beta_i}) \in T$ for $i = 1, 2, \ldots, r$.*
(b) *For each prime p and non-negative integers α, β such that $(p^\alpha, p^\beta) \in T$, there is a unique non-negative integer $\theta_p(\alpha,\beta) \geq \max\{\alpha,\beta\}$ such that $\psi(p^\alpha, p^\beta) = p^{\theta_p(\alpha,\beta)}$.*
(c) *If $(x,y) \in T$, then*

$$\psi(x,y) = \prod_{i=1}^{r} p_i^{\theta_{p_i}(\alpha_i,\beta_i)}. \tag{2.1}$$

Lemma 2.2 (cf. [8], Theorem 3.2). *Let $T \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ be such that*

(a) *$(1,x) \in T$ for every $x \in \mathbb{Z}^+$.*
(b) *$(x,y) \in T$ if and only if $(y,x) \in T$.*
(c) *If x and y are as given in Lemma 2.1, then $(x,y) \in T$ if and only if $(p_i^{\alpha_i}, p_i^{\beta_i}) \in T$, for $i = 1, 2, \ldots, r$.*

Further, for each prime p and non-negative integers α, β such that $(p^\alpha, p^\beta) \in T$, let $\theta_p(\alpha,\beta)$ be a non-negative integer satisfying

(d) *$\theta_p(\alpha,\beta) \geq \max\{\alpha,\beta\}$.*
(e) *$\theta_p(\alpha,\beta) = 0$ if and only if $\alpha = \beta = 0$.*
(f) *$\theta_p(0,\alpha) = 0$, for every $\alpha \geq 0$.*
(g) *$\theta_p(\alpha,\beta) = \theta_p(\beta,\alpha)$.*
(h) *For non-negative integers α, β, γ and for any prime p, the statements ''$(p^\beta, p^\gamma) \in T, (p^\alpha, p^{\theta_p(\beta,\gamma)}) \in T$'' and ''$(p^\alpha, p^\beta) \in T$ and $(p^{\theta_p(\alpha,\beta)}, p^\gamma) \in T$'' are equivalent; when one of these conditions holds, we have $\theta_p(\alpha, \theta_p(\beta,\gamma)) = \theta_p(\theta_p(\alpha,\beta),\gamma)$.*

If for $(x,y) \in T, \psi(x,y)$ is defined by (2.1), then $(F, +, \psi)$ is a commutative ring with unity e and $f\psi g$ is multiplicative whenever f and g are so.

Lemma 2.3 (cf. [8], Theorem 3.3). *Let T and ψ be as in Lemma 2.2. If f is multiplicative and $S_f(k) \neq 0$ for all $k \in \mathbb{Z}^+$, where $S_f(k) = \sum_{\psi(x,k)=k} f(x)$, then f^{-1} is also multiplicative.*

Remark 2.1. If ψ satisfies the hypothesis of Lemma 2.2 and for each $k \in \mathbb{Z}^+, \psi(x,k) = k$ if and only if $x = 1$, then Lemma 2.3 shows that f^{-1} is multiplicative whenever f is so. We note that the condition "$\psi(x,k) = k$ if and only if $x = 1$" is equivalent to saying that for each prime p, "$\theta_p(\alpha,\beta) = \alpha$ if and only if $\beta = 0$".

3. The ψ-analogue of the identical equation

We prove the following:

Theorem. *Let T, ψ and θ_p be as in Lemma 2.2. Further we assume that for each prime p, we have*

$$\text{(3.1)} \qquad \text{(a)} \quad \theta_p(\alpha,\beta)=\theta_p(\alpha,\gamma) \text{ implies that } \beta=\gamma.$$

$$\text{(3.2)} \qquad \text{(b)} \quad \theta_p(a,b)=\theta_p(x,y) \text{ implies that } x=\theta_p(a,c) \text{ for some } c\geq 0 \text{ or } y=\theta_p(b,d) \text{ for some } d\geq 0.$$

If f is multiplicative, then for any positive integers m and n such that $(m,n)\in T$, we have

$$f(\psi(m,n)) = \sum_{\substack{\psi(a,x)=m\\ \psi(b,y)=n}} f(x)f(y)f^{-1}(\psi(a,b))G(a,b), \tag{3.3}$$

where $G(a,b)$ is as given in (1.3).

Proof. Since $\theta_p(\alpha,0)=\alpha$ for each prime p and $\alpha\geq 0$, the condition (3.1) implies that for each prime $p, \theta_p(\alpha,\beta)=\alpha$ if and only if $\beta=0$. Therefore Remark 2.1 shows that f^{-1} exists and is multiplicative, since f is so.

Let $m=\prod_{i=1}^r p_i^{\alpha_i}$ and $n=\prod_{i=1}^r p_i^{\beta_i}$ where $p_1,p_2,\ldots,p_r$ are distinct primes and $\alpha_1,\alpha_2,\ldots,\alpha_r,\beta_1,\beta_2,\ldots,\beta_r$ are non-negative integers. We have by (2.1),

$$f(\psi(m,n)) = f\left(\prod_{i=1}^r p^{\theta_{p_i}(\alpha_i,\beta_i)}\right) = \prod_{i=1}^r f(p_i^{\theta_{p_i}(\alpha_i,\beta_i)}), \tag{3.4}$$

where we have used the fact that f is multiplicative. Let $H(m,n)$ denote the right-hand side of (3.3). We have

$$\begin{aligned} H(m,n) &= \sum_{\substack{\theta_{p_i}(a_i,x_i)=\alpha_i\\ \theta_{p_i}(b_i,y_i)=\beta_i\\ 1\leq i\leq r}} f\left(\prod_{i=1}^r p_i^{x_i}\right) f\left(\prod_{i=1}^r p_i^{y_i}\right) \\ &\qquad \cdot f^{-1}\left(\prod_{i=1}^r p_i^{\theta_{p_i}(a_i,b_i)}\right) G\left(\prod_{i=1}^r p_i^{a_i}, \prod_{i=1}^r p_i^{b_i}\right) \\ &= \sum_{\substack{\theta_{p_i}(a_i,x_i)=\alpha_i\\ \theta_{p_i}(b_i,y_i)=\beta_i\\ 1\leq i\leq r}} \prod_{i=1}^r f(p_i^{x_i})f(p_i^{y_i})f^{-1}(p^{\theta_{p_i}(a_i,b_i)})G(p_i^{a_i},p_i^{b_i}) \\ &= \prod_{i=1}^r \sum_{\substack{\theta_{p_i}(a_i,x_i)=\alpha_i\\ \theta_{p_i}(b_i,y_i)=\beta_i}} f(p_i^{x_i})f(p_i^{y_i})f^{-1}(p_i^{\theta_{p_i}(a_i,b_i)})G(p_i^{a_i},p_i^{b_i}) \\ &= \prod_{i=1}^r H(p_i^{\alpha_i},p_i^{\beta_i}). \end{aligned} \tag{3.5}$$

In view of (3.4) and (3.5), it is enough to prove (3.3) when $m=p^\alpha$ and $n=p^\beta$ where p is a prime and α and β are non-negative integers such that $(p^\alpha,p^\beta)\in T$.

If $m = 1$ or $n = 1$, (3.3) follows trivially. Hence we may assume that α and β are positive integers. Since p is fixed, we write θ for θ_p. From (3.3) and (1.3), we have

$$\begin{aligned} H(p^\alpha, p^\beta) &= \sum_{\substack{\theta(\alpha_1,\alpha_2)=\alpha \\ \theta(\beta_1,\beta_2)=\beta}} f(p^{\alpha_2}) f(p^{\beta_2}) f^{-1}(p^{\theta(\alpha_1,\beta_1)}) G(p^{\alpha_1}, p^{\beta_1}) \\ &= f(p^\alpha) f(p^\beta) - \sum_{\substack{\theta(\alpha_1,\alpha_2)=\alpha \\ \alpha_1>0}} f(p^{\alpha_2}) \sum_{\substack{\theta(\beta_1,\beta_2)=\beta \\ \beta_1>0}} f(p^{\beta_2}) f^{-1}(p^{\theta(\alpha_1,\beta_1)}). \end{aligned} \tag{3.6}$$

Let $\alpha_1 > 0$ be such that $\theta(\alpha_1, \alpha_2) = \alpha$ for some α_2. We have

$$0 = e(p^{\theta(\alpha_1,\beta)}) = \sum_{\theta(x,y)=\theta(\alpha_1,\beta)} f^{-1}(p^x) f(p^y). \tag{3.7}$$

Let

$$S = \{(x, y) \colon \theta(x, y) = \theta(\alpha_1, \beta)\}, \tag{3.8}$$

$$L_1 = \{(\theta(\alpha_1, \beta_1), \beta_2) \colon \theta(\beta_1, \beta_2) = \beta\}, \tag{3.9}$$

and

$$L_2 = \{(b, \theta(a, \beta)) \colon \theta(a, b) = \alpha_1\}. \tag{3.10}$$

Using the assumptions (3.1) and (3.2), it is not difficult to show that

$$S = L_1 \cup L_2, \qquad L_1 \cap L_2 = \{(\alpha_1, \beta)\} \tag{3.11}$$

and

$$L_1 - \{(\alpha_1, \beta)\} = \{(\theta(\alpha_1, \beta_1), \beta_2) \colon \theta(\beta_1, \beta_2) = \beta, \beta_1 > 0\}. \tag{3.12}$$

From (3.8)–(3.12) and (3.7) it follows that

$$\begin{aligned} 0 &= \sum_{\theta(x,y)=\theta(\alpha_1,\beta)} f^{-1}(p^x) f(p^y) \\ &= \sum_{\substack{\theta(\beta_1,\beta_2)=\beta \\ \beta_1>0}} f^{-1}(p^{\theta(\alpha_1,\beta_1)}) f(p^{\beta_2}) + \sum_{\theta(a,b)=\alpha_1} f^{-1}(p^b) f(p^{\theta(a,\beta)}), \end{aligned}$$

so that

$$\sum_{\substack{\theta(\beta_1,\beta_2)=\beta \\ \beta_1>0}} f(p^{\beta_2}) f^{-1}(p^{\theta(\alpha_1,\beta_1)}) = - \sum_{\theta(a,b)=\alpha_1} f^{-1}(p^b) f(p^{\theta(a,\beta)}). \tag{3.13}$$

From (3.13), we have

$$
\begin{aligned}
&\sum_{\substack{\theta(\alpha_1,\alpha_2)=\alpha\\ \alpha_1>0}} f(p^{\alpha_2}) \sum_{\substack{\theta(\beta_1,\beta_2)=\beta\\ \beta_1>0}} f(p^{\beta_2}) f^{-1}(p^{\theta(\alpha_1,\beta_1)})\\
&\quad = - \sum_{\substack{\theta(\alpha_1,\alpha_2)=\alpha\\ \alpha_1>0}} f(p^{\alpha_2}) \sum_{\theta(a,b)=\alpha_1} f^{-1}(p^b) f(p^{\theta(a,\beta)})\\
&\quad = - \sum_{\theta(\alpha_1,\alpha_2)=\alpha} f(p^{\alpha_2}) \sum_{\theta(a,b)=\alpha_1} f^{-1}(p^b) f(p^{\theta(a,\beta)}) + f(p^\alpha) f(p^\beta)\\
(3.14)\qquad &\quad = - \sum_{\theta(a,\theta(b,\alpha_2))=\alpha} f(p^{\alpha_2}) f^{-1}(p^b) f(p^{\theta(a,\beta)}) + f(p^\alpha) f(p^\beta)\\
&\quad = - \sum_{\theta(a,r)=\alpha} f(p^{\theta(a,\beta)}) \sum_{\theta(b,\alpha_2)=r} f^{-1}(p^b) f(p^{\alpha_2}) + f(p^\alpha) f(p^\beta)\\
&\quad = - \sum_{\theta(a,r)=\alpha} f(p^{\theta(a,\beta)} e(p^r) + f(p^\alpha) f(p^\beta)\\
&\quad = - f(p^{\theta(\alpha,\beta)}) + f(p^\alpha) f(p^\beta).
\end{aligned}
$$

Putting (3.14) into (3.6) we obtain that

$$H(p^\alpha, p^\beta) = f(p^{\theta(\alpha,\beta)}),$$

which proves the theorem.

Remark 3.1. Let A be a regular convolution [7], and $T = \bigcup_{n=1}^{\infty} \{(x, n/x) \colon x \in A(n)\}$. Let $\psi \colon T \to \mathbb{Z}^+$ be defined by $\psi(x,y) = xy$ so that for each prime p and non-negative integers α and β such that $(p^\alpha, p^\beta) \in T, \theta_p(\alpha,\beta) = \alpha + \beta$. It is not difficult to show that the assumptions (2.1) and (3.2) are satisfied. In this case, the identity in (3.3) reduces to (1.4′) which is due to P. Haukkanen (cf. [3, Theorem 1.4.8], $G = \mathbb{Z}^+$) as already mentioned in the introduction. Taking $A(m) =$ the set of all divisors of m for each m in (1.4′), we obtain (1.1), and if $A(m) =$ the set of all unitary divisors of m for each m, in (1.4′), we obtain (1.4).

Remark 3.2. We fix a prime p. We define $\theta_p(0,0) = 0$. If n is a positive integer $\neq 3$, we define $\theta_p(x,y) = n$ if and only if $(x,y) \in \{(0,n),(n,0)\}$. If $n = 3$, we define $\theta_p(x,y) = 3$, if and only if $(x,y) \in \{(0,3),(3,0),(1,1)\}$. For a prime $q \neq p, \theta_q(\alpha,\beta)$ is defined in such a way that θ_q satisfies the conditions (d)–(h) of Lemma 2.2 and the conditions (2.1) and (3.2). The set T and the corresponding function ψ can be defined with the aid of Lemma 2.2. It is not difficult to show that θ_p satisfies the conditions (d)–(g) of Lemma 2.2, (2.1) and (3.2). The function ψ so constructed is clearly different from the function $\psi(x,y) = xy$ and yet satisfies the hypothesis of the theorem.

Remark 3.3. Let $T = \mathbb{Z}^+ \times \mathbb{Z}^+$. For each prime p, let $\theta_p(\alpha,\beta) = \alpha + \beta + \alpha\beta$ for non-negative integers α and β. It is not difficult to show that θ_p satisfies the conditions (d)–(h) of Lemma 2.2 and the condition (2.1). Also θ_p does not satisfy the condition (3.2). Let ψ be defined by (2.1). Lemma 2.2 shows that ψ is multiplicativity preserving and Lemma 2.3 shows that f^{-1} is multiplicative whenever f is so. Let $f(x) = x$ for all $x \in \mathbb{Z}^+$. Taking $\alpha = 1$ and $\beta = 2$ in (3.6), we obtain

$$
\begin{aligned}
H(p,p^2) &= p^3 - \sum_{\substack{\theta_p(\alpha_1,\alpha_2)=1 \\ \alpha_1>0}} p^{\alpha_2} \sum_{\substack{\theta_p(\beta_1,\beta_2)=2 \\ \beta_1>0}} p^{\beta_2} f^{-1}(p^{\alpha_1+\beta_1+\alpha_1\beta_1}) \\
&= p^3 - \sum_{\substack{(\alpha_1+1)(\alpha_2+1)=2 \\ \alpha_1>0}} p^{\alpha_2} \sum_{\substack{(\beta_1+1)(\beta_2+1)=3 \\ \beta_1>0}} p^{\beta_2} f^{-1}(p^{\alpha_1+\beta_1+\alpha_1\beta_1}) \qquad (3.15) \\
&= p^3 - f^{-1}(p^5).
\end{aligned}
$$

We have

$$
\begin{aligned}
0 = e(p^5) &= \sum_{\psi(x,y)=p^5} f(x)f^{-1}(y) = \sum_{\theta_p(a,b)=5} p^a f^{-1}(p^b) \\
&= \sum_{(a+1)(b+1)=6} p^a f^{-1}(p^b) = f^{-1}(p^5) + pf^{-1}(p^2) + p^2 f^{-1}(p) + p^5.
\end{aligned}
$$

Applying a similar procedure, we can prove that

$$f^{-1}(p) = -p \quad \text{and} \quad f^{-1}(p^2) = -p^2,$$

so that

$$f^{-1}(p^5) = p^5 - 2p^3;$$

substituting this into (3.15), we obtain

$$H(p,p^2) = p^5 + 2p^3,$$

while

$$f(p^{\theta_p(1,2)}) = p^{\theta_p(1,2)} = p^5.$$

It follows that the identity (3.3) does not hold when $m = p$ and $n = p^2$. The function ψ in this example is originally due to D. H. Lehmer [5].

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