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ON CERTAIN WEIGHTED PARTITIONS AND FINITE SEMISIMPLE RINGS

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ABSTRACT. Let k be a fixed integer $\geqslant 1$ and define $\tau_k(n) = \sum_{d^k/n} 1$. Thus $\tau_1(n)$ is the ordinary divisor function and $\tau_k(n)$ is the number of kth powers dividing n. We derive the asymptotic behaviour as $n \to \infty$ of $P_k(n)$ defined by

$$\sum_{n=0}^{\infty} P_k(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-\tau_k(n)}.$$

Thus $P_k(n)$ is the number of partitions of n where we recognize $\tau_k(m)$ different colours of the integer m when it occurs as a summand in a partition. The case $k = 2$ is of special interest since the number $f(n)$ of semisimple rings with n elements when $n = q_1^{l_1} q_2^{l_2} \ldots$ is given by $f(n) = P_2(l_1)P_2(l_2) \ldots$.

1. Let k be a fixed integer $\geqslant 1$ and define

$$\tau_k(n) = \sum_{d^k/n} 1.$$

Thus $\tau_1(n)$ is the ordinary divisor function and $\tau_k(n)$ is the number of kth powers dividing n. We shall derive the asymptotic behaviour of $p_k(n)$ defined by

$$\sum_{n=0}^{\infty} p_k(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-\tau_k(n)}. \tag{1.0}$$

Thus $p_k(n)$ is the number of partitions of n where we recognize $\tau_k(m)$ different colours of the integer m when it occurs as a summand in a partition. The case $k = 2$ is of special interest since the number of semisimple rings with n elements $f_2(n)$, when

$$n = p_1^{\rho_1} p_2^{\rho_2} \ldots \tag{1.1}$$

is given by $f_2(n) = p_2(\rho_1)p_2(\rho_2) \ldots$ [1]. Also, when k is large, we expect $p_k(n)$ to approach $p(n)$, the number of ordinary partitions.

A generating function for $\tau_k(n)$ is given by

$$\sum_{n=1}^{\infty} \tau_k(n)n^{-s} = \zeta(s)\zeta(ks). \tag{1.2}$$

LEMMA 1. *If $k > 1$,*

$$\sum_{n=1}^{N} \tau_k(n) = \zeta(k)N + O\{N^{1/k}\}$$

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and

$$\sum_{n=1}^{N} \tau_1(n) = N\log N + (2\gamma - 1)N + O\{N^{1/2}\}$$

where γ is Euler's constant.

PROOF. The case $k = 1$ is classical; see, for example, Theorem 320 in [2, p. 264]. The other cases are similar:

$$\sum_{n<x} \tau_k(n) = \sum_{d^k<x} [x/d^k] = x \sum_{d^k<x} \frac{1}{d^k} + O\{x^{1/k}\} = x\zeta(k) + O\{x^{1/k}\}.$$

Lemma 1 shows that if we let $F_\tau(x) = \Sigma_{n\leqslant x}\tau_k(n)$, then $F_\tau(2x) = O\{F_\tau(x)\}$ as $x \to \infty$.

Let us define the function f_τ for real $x > 0$ by

$$f_\tau(x) = \sum_{n=1}^{\infty} \tau_k(n)e^{-xn}.$$

We define α throughout this paper to be the unique solution of

$$n = \sum_{m=1}^{\infty} \tau_k(m)m(e^{\alpha m} - 1)^{-1}. \tag{1.3}$$

THEOREM 1. *Let m be any fixed integer $\geqslant 3$. Let $k \geqslant 1$ be a fixed integer. Then*

$$p_k(n) = (2\pi B_2)^{-1/2}\exp\left\{\alpha n - \sum_{n=1}^{\infty} \tau_k(n)\log(1 - e^{-\alpha n})\right\}$$

$$\times\left[1 + \sum_{\rho=1}^{m-2} D_\rho + O\{f_\tau^{1-2m/3}(\alpha)\}\right].$$

Here we define $B_\mu = B_\mu(n)$ $(\mu = 2, 3, \ldots)$ by

$$B_\mu = \sum_{m=1}^{\infty} \tau_k(m)m^\mu g_\mu(e^{\alpha m})(e^{\alpha m} - 1)^{-\mu} \tag{1.4}$$

where $g_\mu(x)$ is a certain polynomial (the same as in [3] or the g_μ^* of Roth and Szekeres [4]) of degree $\mu - 1$ and, in particular, $g_1(x) = 1$ and $g_2(x) = x$ so that

$$B_2 = \sum_{m=1}^{\infty} \tau_k(m)m^2e^{dm}(e^{dm} - 1)^{-2}.$$

Finally D_ρ $(\rho = 1, 2, \ldots)$ is defined by

$$D_\rho = B_2^{-6\rho} \sum_{\mu_1=2}^{\infty} \cdots \sum_{\mu_{5\rho}=2}^{\infty} d_{\mu_1 \cdots \mu_{5\rho}} B_{\mu_1}B_{\mu_2} \cdots B_{\mu_{5\rho}},$$

the summation being subject to $\mu_1 + \mu_2 + \cdots + \mu_{5\rho} = 12\rho$, and where the d's are certain numerical constants.

PROOF. It is only necessary to note that the conditions of Theorem 1.1 of [3] hold. For convenience we restate the theorem here in terms of the notation of the present paper. We say that τ_k is a P-function if the integers l such that $\tau_k(l) \neq 0$ do not have a common factor > 1 for all sufficiently large l. Then Theorem 1.1 of [3] says:

Let $\tau_k(n)$ have properties (I) and (II). Suppose that $\tau_k(n)$ is a P-function and that $\min_{\tau_k(l)\neq 0}\tau_k(l) > 0$. Suppose furthermore that

$$\varliminf_{x\to\infty} \frac{\log F_\tau(x)}{\log\log x} > 0.$$

Let m be any fixed integer $\geqslant 2$. Then

$$P_k(n) = (2\pi B_2)^{1/2}\exp\left\{\alpha n - \sum_{l=1}^{\infty}\tau_k(l)\log(1 - e^{\alpha l})\right\}$$
$$\times\left[1 + \sum_{l=1}^{m-2} D_e + O\left\{f_\tau^{1-2m/3}(\alpha)\right\}\right].$$

It is not necessary to define conditions (I) and (II) since it is shown in [3] that they hold when $F_\tau(2x) = O\{F_\tau(x)\}$ holds, which we have seen does hold. It is clear that τ_k is a P-function and, furthermore, $\tau_k(l) \geqslant 1$. Also the last condition of Theorem 1.1 holds by Lemma 1. Theorem 1 now follows immediately.

2. In this section we determine the asymptotic behaviour of $p_k(n)$ in terms of elementary functions. First of all, from the Mellin inversion formula,

$$n = \sum_{m=1}^{\infty}\tau_k(m)m(e^{\alpha m} - 1)^{-1} = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\tau_k(m)me^{-\alpha ml}$$
$$= \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty}\alpha^{-t}\Gamma(t)\zeta(t)\sum_{m=1}^{\infty}\tau_k(m)m^{1-t}\,dt$$

for $\sigma > 2$, $|\arg\alpha| < \pi/2$.

It is well known that (equation (1.2))

$$\sum_{m=1}^{\infty}\tau_k(m)m^{-t} = \zeta(t)\zeta(tk);$$

hence,

$$n = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty}\alpha^{-t}\Gamma(t)\zeta(t)\zeta(t-1)\zeta((t-1)k)\,dt. \tag{2.1}$$

LEMMA 2.1. *Let α be defined by equation* (1.3) *with $k = 1$. Then*

$$\alpha = \frac{\pi}{\sqrt{12}}n^{-1/2}\log^{1/2}n\left[1 + O\left(\frac{\log\log n}{\log n}\right)\right].$$

Let $k \geqslant 2$. Then with

$$b_k = \frac{\Gamma(1 + 1/k)\zeta(1 + 1/k)\zeta(1/k)}{(\zeta(2)\zeta(k))^{1/2+1/2k}2k},$$

$$\alpha = n^{-1/2}(\zeta(2)\zeta(k))^{1/2} + n^{1/2k-1}\sqrt{\zeta(2)\zeta(k)}\, b_k + n^{-1}/8 + O\{n^{-1/2k-1}\}.$$

PROOF. The singularities of $\alpha^{-t}\Gamma(t)\zeta(t)\zeta(t-1)\zeta((t-1)k)$ for $k = 1, 2, \ldots$ with real part of $t \geqslant 0$ are at $t = 0, 1, 2$ and $1 + 1/k$. For $k = 1$ there is a double pole, hence the residue at 2 must be evaluated as

$$\left[(d/dt)\{\alpha^{-t}\Gamma(t)\zeta(t)\} + 2\alpha^{-t}\zeta(t)\Gamma(t)(\zeta(t-1) - 1/(t-2))\right]_{t=2}. \tag{2.2}$$

Let us consider the case $k = 1$ first. From equation (2.1) and equation (2.2) and the relations

$$\Gamma'(1) = -\gamma, \quad [\zeta(s) - 1/(s-1)]_{s=1} = \gamma, \quad \zeta(2) = \pi^2/6,$$

we obtain that

$$n = \frac{\pi^2}{6}\frac{\log(1/\alpha)}{\alpha^2} + O\{\alpha^{-2}\}.$$

The first part of the lemma follows from this.

For $k = 2, 3, \ldots$ we obtain from equation (2.1) that

$$n = \alpha^{-2}\zeta(2)\zeta(k) + \alpha^{-1-1/k}\Gamma\left(1 + \frac{1}{k}\right)\zeta\left(1 + \frac{1}{k}\right)\zeta\left(\frac{1}{k}\right)\Big/ k$$
$$+ \alpha^{-1}\zeta^2(0) + O\{1\}$$

and the second part of the lemma follows routinely from this using the fact that $\zeta(0) = -1/2$.

LEMMA 2.2. *Let $k = 1$. Then*

$$\sum_{m=1}^{\infty} \tau_k(m)\log(1 - e^{-\alpha m}) = \frac{\pi^2}{6\alpha}\log\frac{1}{\alpha} + O\{\alpha^{-1}\}.$$

Let $k = 2, \ldots$; then

$$-\sum_{m=1}^{\infty} \tau_k(m)\log(1 - e^{-\alpha m})$$
$$= \alpha^{-1}\zeta(k)\zeta(2) + \frac{\zeta(1/k)}{k}\zeta\left(1 + \frac{1}{k}\right)\Gamma\left(\frac{1}{k}\right)\alpha^{-1/k}$$
$$+ \frac{1}{4}\log\frac{1}{\alpha} - \frac{(1+k)}{2}\zeta'(0) + O\{\alpha\}.$$

PROOF. We derive as above that

$$-\sum_{m=1}^{\infty} \tau_k(m)\log(1 - e^{-\alpha m}) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \alpha^{-t}\Gamma(t)\zeta(1+t)\zeta(t)\zeta(tk)\, dt,$$

and then proceed as in the proof of Lemma 2.1. (Note $\zeta'(0) = -\frac{1}{2}\log 2\pi$ and $\Gamma(t) = 1/t - \gamma + \ldots$.)

LEMMA 2.3. *Let $k = 1$. Then*

$$B_2 = 2\zeta(2)\alpha^{-3}\log(1/\alpha) + O\{\alpha^{-3}\}.$$

Let $k = 2, 3, \ldots$. Then

$$B_2 = 2\alpha^{-3}\zeta(2)\zeta(k) + O\{\alpha^{-2-1/k}\}.$$

PROOF. Note that

$$\sum_{m=1}^{\infty} \tau_k(m)m^2 e^{\alpha m}(e^{\alpha m} - 1)^{-2} = -\frac{d}{d\alpha}\sum_{m=1}^{\infty} \tau_k(m)(e^{\alpha m} - 1)^{-1}$$

$$= \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \alpha^{-t-1}\Gamma(t)\zeta(t)t\zeta(t-1)\zeta((t-1)k)\,dt$$

and the proof proceeds as in Lemmas 2.1 and 2.2.

From Lemmas 2.1, 2.2, 2.3 and Theorem 1, we now obtain, using the facts that $\zeta'(0) = -\frac{1}{2}\log 2\pi$ and $\zeta(2) = \pi^2/6$,

THEOREM 2.1. *As $n \to \infty$,*

$$\log p_1(n) = \frac{\pi}{\sqrt{3}} n^{1/2}\log^{1/2} n\left[1 + O\left\{\frac{(\log\log n)^2}{\log n}\right\}\right].$$

Let $k = 2, 3, \ldots$. Then as $n \to \infty$,

$$p_k(m) = \exp\left\{2\pi n^{1/2}\left(\frac{\zeta(k)}{6}\right)^{1/2} + \frac{\Gamma(1+1/k)\zeta(1+1/k)\zeta(1/k)}{(\zeta(2)\zeta(k))^{1/2k}} n^{1/2k}\right.$$

$$- \frac{n^{1/k-1/2}}{4k^2}\frac{\Gamma^2(1+1/k)\zeta^2(1+1/k)\zeta^2(1/k)}{(\zeta(k)\zeta(2))^{1/2+1/k}}$$

$$\left. + \left(\frac{1+k}{4}\right)\log 2\pi\right\}$$

$$\times \frac{n^{-5/8}}{2\pi^{1/4}}\left(\frac{\zeta(k)}{6}\right)^{1/8}\left[1 + O\{n^{-1/2k}\}\right].$$

Note one could obtain as many terms in the asymptotic expansion as required. However, we have not discovered a general formula.

COROLLARY *Let $f_2(n)$ denote the number of semisimple rings with $n = p^m$ elements. Then with*

$$A = \exp\left(-\frac{9}{4\pi^4}\Gamma^2(1.5)\zeta^2(1.5)\zeta^2(.5)\right)\pi^{3/5}12^{-1/4}\log^{5/8} p,$$

$$f_2(n) \sim A\log^{-5/8} n\exp\left(\frac{\pi^2}{3}\left(\frac{\log n}{\log p}\right)^{1/2}\right.$$

$$\left. + \frac{6^{1/2}}{\pi}\Gamma(1.5)\zeta(1.5)\zeta(.5)\left(\frac{\log n}{\log p}\right)^{1/4}\right).$$

PROOF. It is only necessary to note that if $n = p^m$ then $f_2(n) = p_2(m)$ (see e.g. (1.1)).

This corollary provides the asymptotic formula suggested by Knopfmacher on p. 23 of [5]. In [5] it is also shown that $p_2(n)$ is the number of nonisomorphic semisimple n-dimensional algebras over the Galois field GF(p^r), p a prime.

This corollary shows that the behaviour of $f_2(n)$ is very irregular, since if $n = p$, a prime, then $f_2(n) = 1$. The average behaviour of $f_2(n)$ was originally discussed by Connell [1]. Recently Knopfmacher [5] showed that

$$\sum_{n \leq x} f_2(n) = \alpha_1 x + \alpha_2 x^{1/2} + O(x^{1/3}\log^2 x)$$

where

$$\alpha_1 = \prod_{rm^2 > 1} \zeta(rm^2) = 2.498\ldots, \quad \alpha_2 = \zeta\left(\tfrac{1}{2}\right) \prod_{rm^2 > 1} \zeta\left(\tfrac{1}{2} rm^2\right).$$

However, Knopfmacher [6, Theorem E] has shown that for any $\varepsilon > 0$ there is an integer $n_0(\varepsilon)$ such that

$$f_2(n) < 6^{\frac{1}{4}(1+\varepsilon)(\log n)/(\log\log n)} \quad \text{for all } n \geqslant n_0(\varepsilon),$$

while

$$f_2(n) > 6^{\frac{1}{4}(1-\varepsilon)(\log n)/(\log\log n)} \quad \text{for infinitely many } n.$$

Moreover,

$$f_2(n) < 6^{\frac{1}{4}(1+\varepsilon)\log\log n} \quad \text{for "almost all" } n,$$

i.e. for all n outside same set of asymptotic denisty zero.

Since any partition of n when a one is added to it gives a partition of $n + 1$, it is clear that $p_k(n)$ is monotonic increasing. Furthermore, one may derive from Theorem 1, in a manner similar to that of Roth and Szekeres [4], that if $p_k^{(l)}(n)$ denotes the lth difference of $p_k(n)$ that $p_k^{(l)}(n) \sim \alpha^l p_k(n)$; hence all the differences of $p_k(n)$ are positive for n sufficiently large. Below we give a short table of values of $p_2(n)$ which are useful for computing $f_2(n)$ and the comparison between the asymptotic and true value for certain values of n.

n	$p_2(n)$	n	$p_2(n)$	n	$p_2(n)$
1	1	11	79	21	1549
2	2	12	115	22	2025
3	3	13	154	23	2600
4	6	14	213	24	3377
5	8	15	284	25	4306
6	13	16	391	26	5523
7	18	17	514	27	7000
8	29	18	690	28	8922
9	40	19	900	29	11235
10	58	20	1197	30	14196

n	True Value of $p_2(n)$	Asymptotic Value of $p_2(n)$
100	231 412 7129	2.55495×10^9
200	261 229 585 686401	2.83594×10^{14}
300	246 910 805 791 4492823	2.65888×10^{18}
400	616 439 413 088 071 894 2607	6.60456×10^{21}
500	645 864 386 271 246 677 988 3980	6.89497×10^{24}

We set $\zeta(\frac{1}{2}) = -1.460$, $\zeta(1.5) = 2.612$ and $\Gamma(1.5) = .8862$ in the asymptotic expression. Since the relative error is $O\{n^{1/4}\}$ we cannot expect a rapid decrease in the relative error. The exact values were computed using the recurrence

$$np_2(n) = \sum_{k=1}^{n} a(k)p_2(n-k) \quad \text{where } a(k) = \sum_{d/k} d\tau_k(d).$$

This recurrence is obtained by taking the logarithmic derivative of equation (1.0) and comparing coefficients.

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