ON CERTAIN WEIGHTED PARTITIONS AND FINITE SEMISIMPLE RINGS

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ABSTRACT. Let $k$ be a fixed integer $\geq 1$ and define $\tau_k(n) = \sum d^k / n$. Thus $\tau_1(n)$ is the ordinary divisor function and $\tau_k(n)$ is the number of $k$th powers dividing $n$. We derive the asymptotic behaviour as $n \rightarrow \infty$ of $P_k(n)$ defined by

$$\sum_{n=0}^{\infty} P_k(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-\tau_k(n)}.$$ 

Thus $P_k(n)$ is the number of partitions of $n$ where we recognize $\tau_k(m)$ different colours of the integer $m$ when it occurs as a summand in a partition. The case $k = 2$ is of special interest since the number $f(n)$ of semisimple rings with $n$ elements when $n = q_1q_2^2 \ldots$ is given by $f(n) = P_2(l_1)P_2(l_2) \ldots$.

1. Let $k$ be a fixed integer $\geq 1$ and define $\tau_k(n) = \sum d^k / n$.

Thus $\tau_1(n)$ is the ordinary divisor function and $\tau_k(n)$ is the number of $k$th powers dividing $n$. We shall derive the asymptotic behaviour of $p_k(n)$ defined by

$$(1.0) \quad \sum_{n=0}^{\infty} P_k(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-\tau_k(n)}.$$ 

Thus $p_k(n)$ is the number of partitions of $n$ where we recognize $\tau_k(m)$ different colours of the integer $m$ when it occurs as a summand in a partition. The case $k = 2$ is of special interest since the number of semisimple rings with $n$ elements $f_2(n)$, when

$$(1.1) \quad n = p_1^2p_2^2 \ldots$$ 

is given by $f_2(n) = p_2(p_1)P_2(p_2) \ldots$ [1]. Also, when $k$ is large, we expect $p_k(n)$ to approach $p(n)$, the number of ordinary partitions.

A generating function for $\tau_k(n)$ is given by

$$(1.2) \quad \sum_{n=1}^{\infty} \tau_k(n)n^{-s} = \zeta(s) \zeta(ks).$$

LEMMA 1. If $k > 1$,

$$\sum_{n=1}^{N} \tau_k(n) = \xi(k)N + O \{ N^{1/k} \}$$

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and
\[ \sum_{n=1}^{N} \tau_1(n) = N \log N + (2\gamma - 1)N + O\left\{ N^{1/2} \right\} \]

where \( \gamma \) is Euler's constant.

**Proof.** The case \( k = 1 \) is classical; see, for example, Theorem 320 in [2, p. 264]. The other cases are similar:

\[ \sum_{n < x} \tau_k(n) = \sum_{d^k < x} \left\lfloor \frac{x}{d^k} \right\rfloor = x \sum_{d^k < x} \frac{1}{d^k} + O\left\{ x^{1/k} \right\} = x\zeta(k) + O\left\{ x^{1/k} \right\}. \]

Lemma 1 shows that if we let \( F_\tau(x) = \sum_{n < x} \tau_k(n) \), then \( F_\tau(2x) = O\left\{ F_\tau(x) \right\} \) as \( x \to \infty \).

Let us define the function \( f_\tau \) for real \( x > 0 \) by

\[ f_\tau(x) = \sum_{n=1}^{\infty} \tau_k(n)e^{-xn}. \]

We define \( \alpha \) throughout this paper to be the unique solution of

\[ n = \sum_{m=1}^{\infty} \tau_k(m)m(e^{am} - 1)^{-1}. \]

**Theorem 1.** Let \( m \) be any fixed integer \( \geq 3 \). Let \( k \geq 1 \) be a fixed integer. Then

\[ p_k(n) = (2\pi B_2)^{-1/2} \exp\left\{ \frac{an}{\sum_{n=1}^{\infty} \tau_k(n)\log(1 - e^{an})} \right\} \times \left[ 1 + \sum_{p=1}^{m-2} D_p + O\left\{ f_\tau^{1-2m/3}(\alpha) \right\} \right]. \]

Here we define \( B_\mu = B_\mu(n) \) (\( \mu = 2, 3, \ldots \)) by

\[ B_\mu = \sum_{m=1}^{\infty} \tau_k(m)m^{\mu}g_\mu(e^{am})(e^{am} - 1)^{-\mu} \]

where \( g_\mu(x) \) is a certain polynomial (the same as in [3] or the \( g_* \)'s of Roth and Szekeres [4]) of degree \( \mu - 1 \) and, in particular, \( g_1(x) = 1 \) and \( g_2(x) = x \) so that

\[ B_2 = \sum_{m=1}^{\infty} \tau_k(m)m^{2}e^{dm}(e^{dm} - 1)^{-2}. \]

Finally \( D_\rho \) (\( \rho = 1, 2, \ldots \)) is defined by

\[ D_\rho = B_2^{-6\rho} \sum_{\mu_1=2}^{\infty} \cdots \sum_{\mu_5=2}^{\infty} d_{\mu_1} \cdots d_{\mu_5} B_{\mu_1} B_{\mu_2} \cdots B_{\mu_5}, \]

the summation being subject to \( \mu_1 + \mu_2 + \cdots + \mu_5 = 12\rho \), and where the \( d's \) are certain numerical constants.

**Proof.** It is only necessary to note that the conditions of Theorem 1.1 of [3] hold. For convenience we restate the theorem here in terms of the notation of
the present paper. We say that \( \tau_k \) is a \( P \)-function if the integers \( l \) such that \( \tau_k(l) \neq 0 \) do not have a common factor > 1 for all sufficiently large \( l \). Then Theorem 1.1 of [3] says:

Let \( \tau_k(n) \) have properties (I) and (II). Suppose that \( \tau_k(n) \) is a \( P \)-function and that \( \min_{l \neq 0} \tau_k(l) > 0 \). Suppose furthermore that

\[
\lim_{x \to \infty} \frac{\log F_\tau(x)}{\log \log x} > 0.
\]

Let \( m \) be any fixed integer \( > 2 \). Then

\[
P_k(n) = (2\pi B_2)^{1/2} \exp \left\{ an - \sum_{l=1}^\infty \tau_k(l) \log(1 - e^{al}) \right\} \times 
\left[ 1 + \sum_{l=1}^{m-2} D_e + O \left( f_{\tau}^{1-2m/3} (\alpha) \right) \right].
\]

It is not necessary to define conditions (I) and (II) since it is shown in [3] that they hold when \( F_\tau(2x) = O \{ F_\tau(x) \} \) holds, which we have seen does hold. It is clear that \( \tau_k \) is a \( P \)-function and, furthermore, \( \tau_k(l) > 1 \). Also the last condition of Theorem 1.1 holds by Lemma 1. Theorem 1 now follows immediately.

2. In this section we determine the asymptotic behaviour of \( p_k(n) \) in terms of elementary functions. First of all, from the Mellin inversion formula,

\[
n = \sum_{m=1}^{\infty} \tau_k(m)m(e^{am} - 1)^{-1} = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \tau_k(m)me^{-aml}
\]

\[
= \frac{1}{2\pi i} \int_{\sigma = -i \infty}^{\sigma = i \infty} \alpha^{-t} \Gamma(t) \xi(t) \sum_{m=1}^{\infty} \tau_k(m)m^{-t} \, dt
\]

for \( \sigma > 2, |\arg \alpha| < \pi/2 \).

It is well known that (equation (1.2))

\[
\sum_{m=1}^{\infty} \tau_k(m)m^{-t} = \xi(t)\xi(ik);
\]

hence,

\[
(2.1) \quad n = \frac{1}{2\pi i} \int_{\sigma = -i \infty}^{\sigma = i \infty} \alpha^{-t} \Gamma(t) \xi(t)\xi(t - 1)\xi((t - 1)k) \, dt.
\]

**Lemma 2.1.** Let \( \alpha \) be defined by equation (1.3) with \( k = 1 \). Then

\[
\alpha = \frac{\pi}{\sqrt{12}} \, n^{-1/2} \log^{1/2} n \left[ 1 + O \left( \frac{\log \log n}{\log n} \right) \right].
\]

Let \( k \geq 2 \). Then with

\[
b_k = \frac{\Gamma(1 + 1/k)\xi(1 + 1/k)\xi(1/k)}{(\xi(2)\xi(k))^{1/2 + 1/2k} 2k},
\]
\[ \alpha = n^{-1/2} (\xi(2) \xi(k))^{1/2} + n^{1/2k-1} \sqrt{\xi(2) \xi(k)} \ b_k + n^{-1/8} + O \left( n^{-1/2k-1} \right). \]

**PROOF.** The singularities of \( \alpha^{-1} \Gamma(t) \xi(t) \xi(t-1) \xi((t-1)k) \) for \( k = 1, 2, \ldots \) with real part of \( t > 0 \) are at \( t = 0, 1 \) and \( 1 + 1/k \). For \( k = 1 \) there is a double pole, hence the residue at 2 must be evaluated as

\[ (d/dt) \left\{ \alpha^{-1} \Gamma(t) \xi(t) \right\} + 2 \alpha^{-1} \xi(t) \Gamma(t) (\xi(t-1) - 1/(t-2)) \bigg|_{t=2}. \]

Let us consider the case \( k = 1 \) first. From equation (2.1) and equation (2.2) and the relations

\[ \Gamma'(1) = -\gamma, \quad [\xi(s) - 1/(s-1)]_{s=1} = \gamma, \quad \xi(2) = \pi^2/6, \]

we obtain that

\[ n = \frac{\pi^2}{6} \frac{\log(1/\alpha)}{\alpha^2} + O \left\{ \alpha^{-2} \right\}. \]

The first part of the lemma follows from this.

For \( k = 2, 3, \ldots \) we obtain from equation (2.1) that

\[ n = \alpha^{-1-k} \xi(2) \xi(k) + \alpha^{-1-k} \Gamma \left( 1 + \frac{1}{k} \right) \xi \left( 1 + \frac{1}{k} \right) \xi \left( \frac{1}{k} \right) / k \]

and the second part of the lemma follows routinely from this using the fact that \( \xi(0) = -1/2. \)

**LEMMA 2.2.** Let \( k = 1. \) Then

\[ \sum_{m=1}^{\infty} \tau_k(m) \log(1 - e^{-am}) = \frac{\pi^2}{6 \alpha} \log \frac{1}{\alpha} + O \left\{ \alpha^{-1} \right\}. \]

Let \( k = 2, \ldots \); then

\[ - \sum_{m=1}^{\infty} \tau_k(m) \log(1 - e^{-am}) \]

\[ = \alpha^{-1-k} \xi(k) \xi(2) + \frac{\xi(1/k)}{k} \xi \left( 1 + \frac{1}{k} \right) \Gamma \left( \frac{1}{k} \right) \alpha^{-1/k} \]

\[ + \frac{1}{4} \log \frac{1}{\alpha} - \frac{(1 + k)}{2} \xi'(0) + O \left\{ \alpha \right\}. \]

**PROOF.** We derive as above that

\[ - \sum_{m=1}^{\infty} \tau_k(m) \log(1 - e^{-am}) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \alpha^{-1} \Gamma(t) \xi(1+t) \xi(t) \xi(tk) \ dt, \]

and then proceed as in the proof of Lemma 2.1. (Note \( \xi'(0) = -\frac{1}{2} \log 2\pi \) and \( \Gamma(t) = 1/t - \gamma + \ldots. \))

**LEMMA 2.3.** Let \( k = 1. \) Then

\[ B_2 = 2 \xi(2) \alpha^{-3} \log(1/\alpha) + O \left\{ \alpha^{-3} \right\}. \]

Let \( k = 2, 3, \ldots. \) Then
\[ B_2 = 2\alpha^{-3/2}(2)\zeta(k) + O(\alpha^{-2-1/k}) \].

**Proof.** Note that
\[
\sum_{m=1}^{\infty} \tau_k(m)m^2 e^{am}(e^{am} - 1)^{-2} = -\frac{d}{d\alpha} \sum_{m=1}^{\infty} \tau_k(m)(e^{am} - 1)^{-1}
= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \alpha^{-t-1} \Gamma(t) \zeta(t) \zeta(t-1) \zeta((t-1)k) \, dt
\]
and the proof proceeds as in Lemmas 2.1 and 2.2.

From Lemmas 2.1, 2.2, 2.3 and Theorem 1, we now obtain, using the facts that \( \zeta'(0) = -\frac{1}{2} \log 2\pi \) and \( \zeta(2) = \pi^2/6 \),

**Theorem 2.1.** As \( n \to \infty \),
\[
\log p_1(n) = \frac{\pi}{\sqrt{3}} n^{1/2} \log^{1/2} n \left[ 1 + O \left( \frac{(\log \log n)^2}{\log n} \right) \right].
\]

Let \( k = 2, 3, \ldots \). Then as \( n \to \infty \),
\[
p_k(m) = \exp \left\{ 2\pi n^{1/2} \left( \frac{\zeta(k)}{6} \right)^{1/2} + \frac{\Gamma(1 + 1/k) \zeta(1 + 1/k) \zeta(1/k)}{(\zeta(2) \zeta(k))^{1/2k}} n^{1/2k} \right. \\
- \frac{n^{1/k-1/2}}{4k^2} \frac{\Gamma^2(1 + 1/k) \zeta^2(1 + 1/k) \zeta^2(1/k)}{(\zeta(k) \zeta(2))^{1/2 + 1/k}} \\
+ \left( \frac{1}{4} \right) \log 2\pi \left\} \\
\times \frac{n^{-5/8}}{2\pi^{1/4}} \left( \frac{\zeta(k)}{6} \right)^{1/8} \left[ 1 + O \left( n^{-1/2k} \right) \right].
\]

Note one could obtain as many terms in the asymptotic expansion as required. However, we have not discovered a general formula.

**Corollary** Let \( f_2(n) \) denote the number of semisimple rings with \( n = p^m \) elements. Then with
\[
A = \exp \left( -\frac{9}{4\pi^4} \Gamma^2(1.5) \zeta^2(1.5) \zeta^2(0.5) \right) \pi^{3/5} 12^{-1/4} \log^{5/8} p,
\]
\[
f_2(n) \sim A \log^{-5/8} n \exp \left( \frac{\pi^2}{3} \left( \frac{\log n}{\log p} \right)^{1/2} \\
+ \frac{6^{1/2}}{\pi} \Gamma(1.5) \zeta(1.5) \zeta(0.5) \left( \frac{\log n}{\log p} \right)^{1/4} \right).
\]

**Proof.** It is only necessary to note that if \( n = p^m \) then \( f_2(n) = p_2(m) \) (see e.g. (1.1)).
This corollary provides the asymptotic formula suggested by Knopfmacher on p. 23 of [5]. In [5] it is also shown that \( p_2(n) \) is the number of nonisomorphic semisimple \( n \)-dimensional algebras over the Galois field \( GF(p') \), \( p \) a prime.

This corollary shows that the behaviour of \( f_2(n) \) is very irregular, since if \( n = p \), a prime, then \( f_2(n) = 1 \). The average behaviour of \( f_2(n) \) was originally discussed by Connell [1]. Recently Knopfmacher [5] showed that

\[
\sum_{n < x} f_2(n) = \alpha_1 x + \alpha_2 x^{1/2} + O \left( x^{1/3} \log^2 x \right)
\]

where

\[
\alpha_1 = \prod_{r m^2 > 1} \zeta(r m^2) = 2.498 \ldots, \quad \alpha_2 = \zeta \left( \frac{1}{2} \right) \prod_{r m^2 > 1} \zeta \left( \frac{1}{2} r m^2 \right).
\]

However, Knopfmacher [6, Theorem E] has shown that for any \( \epsilon > 0 \) there is an integer \( n_0(\epsilon) \) such that

\[
f_2(n) < 6^{1/4 (1+\epsilon)(\log n)/(\log \log n)} \quad \text{for all } n > n_0(\epsilon),
\]

while

\[
f_2(n) > 6^{1/4 (1-\epsilon)(\log n)/(\log \log n)} \quad \text{for infinitely many } n.
\]

Moreover,

\[
f_2(n) < 6^{1/4 (1+\epsilon)\log \log n} \quad \text{for "almost all" } n,
\]

i.e. for all \( n \) outside same set of asymptotic density zero.

Since any partition of \( n \) when a one is added to it gives a partition of \( n + 1 \), it is clear that \( p_k(n) \) is monotonic increasing. Furthermore, one may derive from Theorem 1, in a manner similar to that of Roth and Szekeres [4], that if \( p_k^{(l)}(n) \) denotes the \( l \)th difference of \( p_k(n) \) that \( p_k^{(l)}(n) \sim a^l p_k(n) \); hence all the differences of \( p_k(n) \) are positive for \( n \) sufficiently large. Below we give a short table of values of \( p_2(n) \) which are useful for computing \( f_2(n) \) and the comparison between the asymptotic and true value for certain values of \( n \).
We set \( \xi(\frac{1}{2}) = -1.460 \), \( \xi(1.5) = 2.612 \) and \( \Gamma(1.5) = .8862 \) in the asymptotic expression. Since the relative error is \( O(n^{1/4}) \) we cannot expect a rapid decrease in the relative error. The exact values were computed using the recurrence

\[
n p_2(n) = \sum_{k=1}^{n} a(k) p_2(n - k) \quad \text{where} \quad a(k) = \sum_{d|k} d \tau_k(d).
\]

This recurrence is obtained by taking the logarithmic derivative of equation (1.0) and comparing coefficients.

**References**


