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A SIMPLE PROOF OF THE QUINTUPLE PRODUCT IDENTITY

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ABSTRACT. We show here that the important Watson-Gordon five product combinatorial identity can, in fact, be deduced as a very simple and natural corollary to the classical Jacobi triple product identity.

1. **Introduction.** The following fundamental identity is of great importance in combinatorial analysis:

$$\begin{aligned}(1.1)\quad &\prod_{n=1}^{\infty}(1-s^n)(1-s^nt)(1-s^{n-1}t^{-1})(1-s^{2n-1}t^2)(1-s^{2n-1}t^{-2})\\ &\qquad = \sum_{n=-\infty}^{\infty} s^{(3n^2+n)/2}(t^{3n}-t^{-3n-1}), \qquad |s|<1,\ t\neq 0.\end{aligned}$$

This identity, whose origin may be traced to an elliptic sigma formula of Weierstrass, has a very interesting history which we shall give in the last section.

In this paper, we shall derive this identity as a simple and natural corollary of Jacobi's triple product identity:

$$(1.2)\quad \prod_{n=1}^{\infty}(1-x^{2n})(1-ax^{2n-1})(1-a^{-1}x^{2n-1}) = \sum_{n=-\infty}^{\infty}(-1)^n a^n x^{n^2},$$

where $|x|<1$ and $a\neq 0$. For the simplest and most elementary proof of (1.2), we refer to George Andrews **[1]**.

Throughout this note, $\sum_n$ denotes summation from $n=-\infty$ to $n=\infty$, while $\prod_n$ denotes product from $n=1$ to $n=\infty$.

2. **Proof of (1.1).** The identity (1.1) may be written in the form

$$\begin{aligned}(2.1)\quad &\prod_n (1-s^{2n})(1-s^{2n}t)(1-s^{2n-2}t^{-1})(1-s^{4n-2}t^2)(1-s^{4n-2}t^{-2})\\ &\qquad = \sum_n s^{n(3n+1)}(t^{3n}-t^{-3n-1}).\end{aligned}$$

Let $A(s,t)$ denote $\prod(1-s^{4n})$ times the left number of (2.1). Applying

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(1.2) twice, we see that

$$\begin{aligned} A(s,t) &= \sum_j (-1)^j s^{j^2+j} t^j \sum_k (-1)^k s^{2k^2} t^{2k} \\ &= \sum_{i,k} (-1)^{j+k} s^{j^2+2k^2+j} t^{j+2k}. \end{aligned}$$

Writing $j+2k=n$, we get

$$\begin{aligned} A(s,t) &= \sum_n (-1)^n t^n \sum_k (-1)^k s^{(n-2k)^2+2k^2+n-2k} \\ &= \sum_n (-1)^n t^n \sum_k (-1)^k s^{n^2+6k^2-4nk+n-2k} \qquad (2.2) \\ &= \sum_n (-1)^n t^n s^{n^2+n} \sum_k (-1)^k s^{6k^2-4nk-2k}. \end{aligned}$$

Now, for all integers p,

$$\sum_k (-1)^k s^{6k^2+6k(2p+1)} = 0.$$

This follows on replacing k by $-k-2p-1$. Hence the inner sum in (2.2) vanishes unless $-4n-2 \not\equiv 0 \pmod 6$, that is, $2n+1 \not\equiv 0 \pmod 3$; so that we can assume in (2.2) that $n \equiv 0$ or $-1 \pmod 3$.

Let $A_1(s,t)$ and $A_2(s,t)$ denote respectively the parts of $A(s,t)$ corresponding to $n \equiv 0 \pmod 3$ and $n \equiv -1 \pmod 3$. Then

$$A_1(s,t) = \sum_n (-1)^n t^{3n} s^{9n^2+3n} \sum_k (-1)^k s^{6k^2-12nk-2k}.$$

Write $m=k-n$ in the inner series. We get, after a routine simplification,

$$A_1(s,t) = \sum_n t^{3n} s^{3n^2+n} \sum_n (-1)^m s^{6m^2-2m} = Q \sum_n t^{3n} s^{3n^2+n} \qquad (2.3)$$

where $Q = \prod_m (1 - s^{4m}) = \sum_m (-1)^m s^{6m^2-2m}$. Similarly, we get

$$\begin{aligned} A_2(s,t) &= Q \sum_n (-1)^{n-1} t^{3n-1} s^{3n^2-n} \qquad (2.4) \\ &= Q \sum_n (-1)^{n-1} t^{-3n-1} s^{3n^2+n}. \end{aligned}$$

Combining (2.3) and (2.4) we obtain (2.1), and hence (1.1).

3. **A historical note.** In 1929, G. N. Watson **[11]** derived the identity (1.1) in the course of proving some of Ramanujan's theorems on continued fractions. In 1938, Watson **[12]** proved the following identity:

$$\begin{aligned} \prod_n \frac{(1-x)^{2n}(1-a^2x^{2n-2})(1-a^{-2}x^{2n})}{(1+ax^{2n-1})(1+a^{-1}x^{2n-1})} \qquad (3.1) \\ = \sum_n (a^{-3n} - a^{3n+2}) x^{n(3n+2)}, \qquad |x| < 1,\ a \neq 0. \end{aligned}$$

As was pointed out in **[10]**, a simple transformation shows that (1.1) and (3.1) are equivalent. Another proof of (3.1), using function-theoretic methods, was given in 1954 by A. O. L. Atkin and P. Swinnerton-Dyer **[2**, Lemma 5]. In 1951, W. N. Bailey **[3]** obtained a proof of (1.1), but acknowledged the priority of Watson. In 1952, D. B. Sears [**8**] deduced the identity (1.1) from a result of his that is essentially the 3-term sigma function relation due to Weierstrass. L. J. Slater [**9**, pp. 204–205, especially (7.4.7)] gives the details of the deduction from the sigma function relation.

In 1961, Basil Gordon [**6**] independently discovered (1.1) and gave several important applications of the same. L. J. Mordell [**7**] presumed that Basil Gordon is the original discoverer of (1.1) and supplied a new proof of this identity. Finally, Carlitz [**4**] gave a proof of (1.2) based on entirely different ideas.

In view of the above history of the identity (1.1), we agree with the referee that it is best to refer to the identity (1.1) simply as the quintuple product identity, or, if we wish to use proper names, as the Weierstrass-Watson identity.

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Finally, we wish to remark that the method used here to prove (1.1) is applicable for some other identities also, as shown in [**5**].

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