PERFECT TRIANGLES

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It was proved in 1904 by W. A. Whitworth and D. Biddle (see L. E. Dickson, *History of the Theory of Numbers*, vol. II, p. 199) that the only triangles (a, b, c) the sum of whose integer-valued sides equals the area of the triangle are (5, 12, 13), (6, 8, 10), (6, 25, 29), (7, 15, 20), and (9, 10, 17). In 1955, R. R. Phelps defined a perfect triangle (a, b, c) as one whose integer-valued sides add to twice its area, and proposed the problem of finding all such triangles (this MONTHLY, 62 (1955) p. 365). N. J. Fine, who provided the solution (this MONTHLY, 63 (1956) pp. 43–44) showed that there is only one such triangle, namely (3, 4, 5).

One can analogously ask if there is a triangle (a, b, c) whose integer-valued sides add up to thrice the area. It can be shown that such a triangle does not exist. Actually, we shall prove a more general result.

THEOREM 1. *Let $N(\lambda)$ denote the number of triangles (a, b, c) whose integer-valued sides a, b, c add up to λ times their area. Then $N(\lambda)$ is finite for all positive values of λ; moreover, $N(\lambda)=0$ for all $\lambda>\sqrt{8}$ with the exception $N(2\sqrt{3})=1$ (in which case the triangle is $(2, 2, 2)$).*

Proof. The area A of the triangle (a, b, c) is given by

$$A^2 = s(s-a)(s-b)(s-c), \tag{1}$$

where $2s=a+b+c$. Let us assume that $2s=\lambda A$, $\lambda>0$. Write $\lambda/2=\mu$ and let

$$X = \mu A - a, \quad Y = \mu A - b, \quad Z = \mu A - c. \tag{2}$$

Then (1) gives $XYZ=A/\mu$. But $X+Y+Z=\mu A$, so that

$$X + Y + Z = \mu^2 XYZ. \tag{3}$$

Now, X, Y, Z are positive, and we shall assume (as we may) that $X \geqq Y \geqq Z$. Hence,

$$\begin{aligned} Y(YZ-1) \leqq X(YZ-1) &= XYZ - X = A/\mu - X \\ &= (X+Y+Z)\mu^{-2} - X = \mu^{-2}(Y+Z) - (1-\mu^{-2})X, \end{aligned}$$

giving

$$Y(YZ-1) < \begin{cases} \mu^{-2}(Y+Z), & \mu > 1 \\ (2\mu^{-2}-1)(Y+Z), & \mu \leqq 1, \end{cases} \tag{4}$$

where, for the case $\mu<1$, we use the relation $X<Y+Z$ (which is a consequence of the "triangle inequality"). Taking first the case $\mu>1$ and using our convention that $Z \leqq Y$, we see from (4) that

$$YZ < 1 + \frac{2}{\mu^2}, \tag{5}$$

and so

$$Z^2 < 1 + \frac{2}{\mu^2} \cdot \tag{6}$$

Since X, Y, Z are positive integers, inequality (6) implies that Z can take only a finite number of values. The same holds for Y (on using (5)) and X (on using (3) or the relation $X < Y+Z$).

Since $X+Y+Z=\mu A$, A takes only a finite number of values. So do the sides a, b, c, in view of relations (2). Thus $N(\lambda)$ is finite for each positive $\lambda > 2$.

If $\lambda \leqq 2$ (i.e., $\mu \leqq 1$), we obtain the same conclusion on using the second inequality in (4) and proceeding as before.

We shall next show that $N(\lambda)=0$ for $\lambda > \sqrt{8}$ (i.e., $\mu > \sqrt{2}$). For such a value of μ we have from (6) that $Z=1$. The relation (5) then shows that $Y=1$. Since $X < Y+Z$, we now have $X=1$. Using relations (3) and (2), one obtains $\mu=3$, $A=\sqrt{3}$, $a=b=c=2$.

REMARKS. The remarks made at the beginning of this note show that $N(1)=5$ and $N(2)=1$. Our theorem shows that $N(3)=N(4)= \cdots =0$.

It would be interesting to consider a similar problem for a quadrilateral, and in general, a polygon of n sides ($n>2$).

RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Material should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary 44, Alberta, Canada.

HOW OFTEN DOES AN INTEGER OCCUR AS A BINOMIAL COEFFICIENT?

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Let $N(a)$ be the number of times a occurs as a binomial coefficient, $\binom{n}{k}$. We have $N(1)=\infty$, $N(2)=1$, $N(3)=N(4)=N(5)=2$, $N(6)=3$, etc. Clearly, for $a>1$, $N(a)<\infty$. Below we establish that $N(a)=O(\log a)$. We conjecture that $N(a)=O(1)$, that is, that the number of solutions of $\binom{n}{k}=a$ is bounded for $a>1$. Erdös, in a private communication, concurs in this conjecture and states that it must be very hard. In a later communication, he suggests trying to show $N(a)=O(\log \log a)$.

If we let $M(k)$ be the first integer a such that $N(a)=k$, we have: $M(1)=2$, $M(2)=3$, $M(3)=6$, $M(4)=10$, $M(6)=120$. The next values would be interesting to know.