where the columns in $W^{(k)}(t)$ are those in $W(t)$, except for the kth which is replaced by its derivative. Then if $\Phi(t)=(X_1X_2\cdots X_k\cdots X_n)$

$$\begin{aligned} W^{(k)}(t) &= |X_1X_2\cdots X_k'\cdots X_n| \\ &= |X_1X_2\cdots AX_k\cdots X_n|. \end{aligned} \tag{21}$$

From (19) we observe that the columns $X_k(t)$ are unit vectors such that

$$X_j(\tau) = \begin{bmatrix} \delta_{1j} \\ \delta_{2j} \\ \cdot \\ \cdot \\ \cdot \\ \delta_{nj} \end{bmatrix} \qquad \text{where } \delta_{ij} = 0, \quad i \neq j \\ \qquad\qquad = 1, \quad i = j.$$

Then

$$A(\tau)X_k(\tau) = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \cdot \\ \cdot \\ \cdot \\ a_{nk} \end{bmatrix} \tag{22}$$

Then (21) and (22) combined show that $W^{(k)}(\tau)=a_{kk}(\tau)$. The latter coupled with (20) and (18) shows that

$$f(\tau) = \frac{W'(\tau)}{W(\tau)} = \sum_{k=1}^{n} a_{kk}(\tau) = \operatorname{tr} A(\tau).$$

Since τ is an arbitrary point $W'(t)=f(t)W(t)=\operatorname{tr} A(t)W(t)$ which leads us back to (5).

This work was supported in part by the National Science Foundation under grant GP-4171.

References

1. H. Hochstadt, Differential Equations—A Modern Approach, Holt, Rinehart and Winston, New York, 1964.
2. F. John, Ordinary Differential Equations, Lecture Notes: New York University, 1965.
3. L. S. Pontryagin, Ordinary Differential Equations, Addison-Wesley, Reading, Mass, 1962.

A SIMPLE IRRATIONALITY PROOF FOR QUADRATIC SURDS

M. V. Subbarao, University of Missouri

Let N be a positive integer which is not a square of another integer. If $\sqrt{N}$ is rational, we will obtain contradictions in three ways, thus providing three different proofs for the irrationality of $\sqrt{N}$.

Write $\sqrt{N}=a/b$, where the fraction on the right is chosen so that:

1. The numerator a is the smallest possible positive integer (for the first proof);

2. The denominator b is the smallest possible positive integer (for the second proof);

3. The sum of the numerator and denominator is the smallest possible (for the third proof).

Since $a^2=Nb^2$ we have $a^2-Aab=Nb^2-Aab$, where A is the unique positive integer given by $A<\sqrt{N}<A+1$. Hence, $a(a-Ab)=b(Nb-Aa)$ giving $\sqrt{N}=a/b=(Nb-Aa)/(a-Ab)$. But, in this new expression for $\sqrt{N}$, the numerator $Nb-Aa$ is less than a, the denominator $a-Ab$ is less than b, and the sum of the numerator and denominator is less than $a+b$—three contradictions to complete the three proofs!

Whether this kind of reasoning can be extended to establish the irrationality of the kth root of a non-kth power integer for $k>2$ is an open question; the writer's attempts in this direction ran into difficulties.

LEAST SQUARES LINE BY GRAPHICAL METHOD

J. B. WILSON, North Carolina State University

1. Introduction. To find the best fitting line $y=ax+b$ for a set of points (x_k, y_k) of equal weight, $k=0, 1, \cdots, n$, according to the theory of least squares, one determines the coefficients a and b from the normal equations

$$\begin{aligned} \sum_{k=0}^{n} y_k &= a\sum_{k=0}^{n} x_k + (n+1)b \\ \sum_{k=0}^{n} x_k y_k &= a\sum_{k=0}^{n} x_k^2 + b\sum_{k=0}^{n} x_k. \end{aligned} \tag{1}$$

The graphical procedure discussed below for determining this line has been in use for some time, but apparently is not widely known. The procedure and its justification are felt to be of classroom interest.

2. Procedure. For convenience, the abscissa x_0 is taken to be zero, there being no loss of generality. The difference x_k-x_{k-1}, assumed to be constant, is denoted by h, and the given points (x_k, y_k) are denoted by P_k.

Let P_1' denote the point on the segment P_0P_1 with abscissa $2h/3$, P_2' the point on $P_1'P_2$ with abscissa $4h/3$, and, in general, P_k' the point on $P_{k-1}'P_k$ with abscissa $2kh/3$. The final such point is the point A (or P_n') on $P_{n-1}'P_n$ with abscissa $2nh/3$. A point B is located by a similar "two-thirds" procedure, but beginning at P_n, instead of P_0, and proceeding to the left.

The line drawn on points A and B is the required least squares line. The adjoining figure illustrates the procedure for finding point A in the case of five points. For this case we would have

$$A(8h/3, (y_0+2y_1+3y_2+4y_3+5y_4)/15)$$