(d) If $c \geqq 0$, then $F(x) > 0$ for all x and so the curve $v^2 = F(x)$ does not cut the x-axis. That is, no solution is oscillatory.

We summarize these possibilities in the following theorem.

THEOREM 1. *Let* $c = \exp(-x_0^2)[x_0'^2 - q^2(x_0^2+1)]$. *A solution of* (1) *with initial conditions* $x = x_0$, $x' = x_0'$ *and valid for all large* t *is oscillatory if* $-q^2 < c < 0$ *and nonoscillatory if* $c \geqq 0$.

The analysis of equation (2) is similar. The equation corresponding to (6) is

$$v^2 = z = q^2(x^2 - 1) + ce^{-x^2}. \tag{7}$$

The curve $v^2 = F(x) = 0$ for equation (7) may be described as follows. It is similar to a parabola symmetric about the c axis, opening downward with intercepts at $(\pm 1, 0)$, and having $(0, q^2)$ as highest point.

For no value of c (except the trivial case $c = q^2$) is (7) a closed curve and so we have the following theorem.

THEOREM 2. *No solution of equation* (2) *is oscillatory.*

A similar analysis of (3) and (4) reveals, as was shown by Utz, that all solutions of (3), $x \not\equiv 0$, valid for all large t are oscillatory while all of the solutions of (4) are nonoscillatory.

The author wishes to thank the referee for helpful suggestions.

Reference

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ARITHMETIC FUNCTIONS AND DISTRIBUTIVITY

M. V. SUBBARAO, University of Alberta

1. Introduction. In a recent paper Lambek [2, Theorem 2] proved the equivalent of the following theorem:

Let f, g, h, k be completely multiplicative functions, then

$$(f \circ g)(h \circ k) = fh \circ fk \circ gh \circ gk \circ w, \tag{1.1}$$

where $w(n) = f(\sqrt{n})\, g(\sqrt{n})\, h(\sqrt{n})\, k(\sqrt{n})\, \mu(\sqrt{n})$ if n is a square and $w(n) = 0$ otherwise and $\mu(n)$ is the Moebius function. Here and throughout what follows " $\circ$ " denotes Dirichlet convolution. A function f is called multiplicative (completely multiplicative) provided $f(1) = 1$ and $f(mn) = f(m)\, f(n)$ for all coprime integers m and n (for all positive integers m and n).

In this note, we give a very short proof of (1.1) different from Lambek's and then extend (1.1) to the case of triple product $(f \circ g)\,(h \circ k)\,(u \circ v)$, where all the functions $f, \cdots, v$ involved are completely multiplicative. As a corollary we obtain the identity:

(1.2) $\sum \sigma_\alpha(n)\sigma_\beta(n)\sigma_\gamma(n)/n^s$

$$= \zeta(s)\zeta(s-\alpha)\zeta(s-\beta)\zeta(s-\gamma)\zeta(s-\alpha-\beta)\zeta(s-\beta-\gamma)\zeta(s-\gamma-\alpha) \cdot \zeta(s-\alpha-\beta-\gamma)\theta(s),$$

where $\zeta(s)$ is the Riemann Zeta function, $\sigma_\alpha(n)$ represents the sum of the αth powers of the divisors of n, and $\theta(s) = \sum F(n)/n^s$, $F(n)$ being a multiplicative function of n defined, for arbitrary prime p, by $F(1) = 1$; $F(p^m) = 0$ for $m = 1$ or 5 or $m > 6$;

$$F(p^4) = p^{\alpha+\beta+\gamma}F(p^2) = -p^{\alpha+\beta+\gamma}(p^{\alpha+\beta+\gamma}(p^\alpha + p^\beta + p^\gamma + 3) + p^{\beta+\gamma} + p^{\gamma+\alpha} + p^{\alpha+\beta})$$

and

$$F(p^3) = p^{\alpha+\beta+\gamma}(p^\alpha + 1)(p^\beta + 1)(p^\gamma + 1); \quad F(p^6) = -p^{3(\alpha+\beta+\gamma)}.$$

On lettering $\gamma \to -\infty$, (1.2) reduces to a well-known result of Ramanujan [**3**].

Later in this note we consider the case of unitary convolution and show that the distribution law analogous to (1.1) holds 'exactly' without an 'error term' like w in (1.1). In fact, if f, g, h, k are arbitrary *multiplicative* functions, then

$$(f \cdot g)(h \cdot k) = fh \cdot fk \cdot gh \cdot gk, \tag{1.3}$$

where '$\cdot$' denotes the unitary convolution operation defined by

$$(f \cdot g)(n) = \sum_{\substack{d \mid n \\ (d, n/d) = 1}} f(d)g(n/d). \tag{1.4}$$

If S denotes the set of all multiplicative functions, and X denotes the natural product of two such functions, the ring $(S, \cdot, X)$ has some interesting properties which will be dealt with elsewhere.

2. Dirichlet convolution and distributivity. To prove (1.1) set $f(p) = a_p$, $g(p) = b_p$, $h(p) = c_p$, $k(p) = d_p$, $p^{-s} = x_p$ where p is an arbitrary prime. Then using formal Dirichlet series we have

$$\sum (f \circ g)(n)/n^s = \prod_p (1 - a_p x_p)^{-1}(1 - b_p x_p)^{-1}, \tag{2.1}$$

where throughout the paper, $\sum$ denotes summation over all positive integers n and $\prod_p$ denotes product over all primes p. Thus,

$$\sum (f \circ g)(h \circ k)(n)/n^s = \prod_p \left\{1 + \sum \frac{a_p^{n+1} - b_p^{n+1}}{a_p - b_p}\right\} \left\{\frac{c_p^{n+1} - d_p^{n+1}}{c_p - d_p}\right\} x_p^n. \tag{2.2}$$

(If $b_p = a_p$ in the above, $[a_p^{n+1} - b_p^{n+1}]/[a_p - b_p]$ is to be taken as its limiting value $(n+1)a_p^n$ as $b_p \to a_p$; and similarly for $[c_p^{n+1} - d_p^{n+1}]/[c_p - d_p]$.) The result (1.1) now follows immediately from the elementary identity

$$(2.3) \qquad 1+\sum \frac{a^{n+1}-b^{n+1}}{a-b}\,\frac{c^{n+1}-d^{n+1}}{c-d}\,x^n=\frac{1-abcdx^2}{(1-acx)(1-adx)(1-bcx)(1-bdx)}$$

on noticing that $\sum w(n)/n^s=\prod_p (1-abcdx^2)$.

We shall next extend the result (1.1) to the triple product $(f \circ g)(h \circ k)(u \circ v)$, where all the functions $f, g, \cdots, v$ are completely multiplicative. Set, as above, $f(p)=a_p, \cdots, k(p)=d_p, u(p)=i_p, v(p)=j_p, p^{-s}=x_p$, and use the identity

$$(2.4) \qquad \begin{aligned} &1+\sum \left\{\frac{a^{n+1}-b^{n+1}}{a-b}\right\}\left\{\frac{c^{n+1}-d^{n+1}}{c-d}\right\}\left\{\frac{i^{n+1}-j^{n+1}}{i-j}\right\}x^n \\ &=\frac{1-rx^2+2abcdij(a+b)(c+d)(i+j)x^3-abcdijrx^4+(abcdij)^3x^6}{(1-acix)(1-acjx)(1-adix)(1-adjx)(1-bcix)(1-bcjx)(1-bdix)(1-bdjx)} \end{aligned}$$

where

$$(2.5) \qquad r = (a^2+b^2)cdij + (c^2+d^2)abij + (i^2+j^2)abcd + 3abcdij.$$

We thus obtain the following

THEOREM 1. *For arbitrary completely multiplicative functions f, g, h, k, u, v, we have*

(2.6) $(f\circ g)(h\circ k)(u\circ v)=fhu\circ fhv\circ fku\circ fkv\circ ghu\circ ghv\circ gku\circ gkv\circ t$, where $t(n)$ is a multiplicative function defined for arbitrary primes p as follows:

$$t(p^m)=\begin{cases} 1 & \text{if } m=0; \\ 0 & \text{if } m=1 \text{ or } 5 \text{ or } m>6; \\ -r & \text{if } m=2; \\ 2abcdij(a+b)(c+d)(i+j) & \text{if } m=3; \\ -abcdijr & \text{if } m=4; \\ (abcdij)^3 & \text{if } m=6, \end{cases}$$

where r is given by (2.5).

Setting $f(n)=n^\alpha$, $h(n)=n^\beta$; $j(n)=n^\gamma$; $g(n)=k(n)\equiv 1$, we obtain the result (1.2) stated earlier.

One can easily see that the general situation is as follows: If $f_{i1}, f_{i2} (i=1, \cdots, r)$ are all completely multiplicative, then the natural product of the r functions $f_{i1}\circ f_{i2}$ $(i=1,\cdots,r)$ equals the Dirichlet convolute of the 2^r functions $f_{1j}f_{2j}\cdots f_{rj}$ $(j=1, 2)$ and a certain multiplicative function $t_r(n)$ which vanishes whenever the canonical form of n has a prime occurring to an exponent <2 or $>2^r-2$.

3. Unitary convolution and distributivity. We shall first note the

LEMMA. *The arithmetic function f satisfies $f(g\cdot h)=fg\cdot fh$ for all arithmetic functions g and h if and only if h is multiplicative.*

Here '·' denotes the unitary convolution defined in (1.4). The 'if' part of the proof is quite trivial. For the 'only if' part, given any integer M, define

$$\delta_M(n) = \begin{cases} 1 & \text{if } n = M, \\ 0 & \text{otherwise.} \end{cases}$$

Then, evaluating $f(\delta_M \cdot \delta_N) = f_M \cdot f_N$ at $n = MN$, one obtains $f(MN) = f(M)f(N)$ as required.

THEOREM 2. *For any multiplicative functions f, g, h and k, we have*

$$(f\cdot g)(h\cdot k) = fh\cdot fk\cdot gh\cdot gk.$$

Proof. Since f and g are multiplicative, so also is $f\cdot g$. Application of the lemma yields

$$(f\cdot g)(h\cdot k) = [(f\cdot g)h]\cdot[(f\cdot g)k].$$

Applying the lemma again, the right member in the above relation becomes $(fh\cdot gh)\cdot(fk\cdot gk)$, thus proving the theorem.

4. Concluding remarks. It is well known (see, for example, [1]) that if a_n and b_n are sequences, each of which satisfies a linear recurrence relation with constant coefficients, the sequence a_nb_n satisfies a similar relation. Moreover, if a_n is generated by the function

$$A(x) = \sum a_n x^n = \frac{p(x)}{(1-\theta_1 x)^{e_1}\cdots(1-\theta_k x)^{e_k}},$$

where the θ_i are distinct numbers and $p(x)$ a polynomial in x of degree $< e_1 + e_2 + \cdots + e_k = N$ and if b_n is generated by $B(x)$, then the sequence a_nb_n is generated by

$$C(x) = \sum a_n b_n x^n = \sum_{j=1}^{k} \frac{1}{(e_j-1)!}\frac{\partial^{e_j-1}}{\partial s^{e_j-1}} \cdot \left\{\frac{s^{N-1}p(x/s)B(s)}{(s-\theta_1 x)^{e_1}\cdots(s-\theta_{j-1}x)^{e_{j-1}}(s-\theta_{j+1}x)^{e_{j+1}}\cdots}\right\}_{s=\theta_j}$$

Using this method one easily obtains (2.3) and (2.4) and many others. It can be used for example to obtain an identity for $\sum\sigma_\alpha(n)\sigma_\beta(n)\sigma_\gamma(n)\sigma_\delta(n)/n^s$ similar to (1.2). The details are left to the interested reader. One can similarly obtain identities involving other arithmetic functions also which satisfy a linear recurrence relation with constant coefficients. As simple illustrations we might mention:

(4.1) $$\sum \sigma(n)\phi(n)/n^s = \zeta(s-1)\zeta(s-2)\prod_p (1-(p+1)p^{-s}+p^2p^{-2s}); \quad s>2;$$

and

(4.2) $$\sum \overset{*}{\sigma}_\alpha(n)\overset{*}{\sigma}_\beta(n)/n^s = \zeta(s)\zeta(s-\alpha)\zeta(s-\beta)\zeta(s-\alpha-\beta)F(s),$$

where

$$F(s) = \prod_p \{1 + (p^\alpha + p^\beta + p^{2\alpha+\beta} + p^{\alpha+2\beta} + 2p^{\alpha+\beta})p^{-2s} + p^{\alpha+\beta}(1 + p^\alpha)(1 + p^\beta)p^{-3s} - 3p^{2\alpha+2\beta}p^{-4s}\}.$$

Here $\phi(n)$ denotes the Euler totient; $\sigma(n) = \sigma_1(n)$; and

$$\overset{*}{\sigma}_\alpha(n) = \sum_{\substack{d\mid n \\ (d,n/d)=1}} d^\alpha = \text{the sum of the } \alpha\text{th powers of the unitary divisors of } n.$$

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BRIEF VERSIONS

Because of the extraordinary pressure for publication, some papers are being presented in brief form in this department of the MONTHLY. *Authors have agreed to provide interested readers with extended versions of their papers. The address to which to write for such an extended version is given at the end of each paper.*

THE ARITHMETIC FUNCTION $\tau_{k,r}(n)$

R. SIVARAMAKRISHNAN, Government Engineering College, Trichur, India

1. We define $\tau_{k,r}(n)$ as the number of ways of expressing n as the product of k factors each of which is an rth power including unity ($r \geqq 1$), regarding factorizations as distinct, according to the order of the factors. The special case for $r=1$ is $\tau_k(n)$ (vide [1]).

From the definition it can be easily shown that

(1.1) $\tau_{k,r}(n)$ is multiplicative in n

(1.2) $$\tau_{k,r}(n) = \begin{cases} \tau_k(n^{1/r}), & \text{if } n \text{ is an } r\text{th power} \\ 0, & \text{otherwise.} \end{cases}$$

2. Using (1.1) and (1.2), we prove the following theorems:

(2.1) $$\sum_{d\mid n} \tau_{k+1,r}(d)\phi_r\left(\frac{n}{d}\right) = \sum_{d\mid n} d\tau_{k,r}\left(\frac{n}{d}\right),$$

where $\phi_r(n)$ is the extension of Euler's ϕ-function due to V. L. Klee [2].