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A CLASS OF ADDITIVE FUNCTIONS

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1. Introduction. Throughout this note, let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the representation of $n > 1$ as a product of powers of distinct primes, and define $\Omega_k(n) = a_1^k + \cdots + a_r^k$; $w(n) = \Omega_0(n)$; $\Omega(n) = \Omega_1(n)$. In a series of interesting papers ([1]–[5]) R. L. Duncan considered these functions $\Omega_k(n)$ and in particular obtained identities ([4], Theorem 1) which generalize some results in Titchmarsh ([6] Ch. 1, eqs 1.6.2 and 1.6.3). In the paper [5] to appear, an advance copy of which he kindly sent me, he considers an even more general class of additive functions, given by

$$(1.1) \quad a(n) = G(a_1) + \cdots + G(a_r) \quad \text{for } n > 1; \quad a(1) = 0.$$

Here $G(n)$ is an arbitrary arithmetic function for which $G(0) = 0$ and $G(n) \geq G(n-1)$ for $n \geq 1$. Duncan establishes the following result: if

$$(1.2) \quad b(n) = \sum_{d|n} \{G(d) - G(d-1)\} d\mu(n/d),$$

then

$$(1.3) \quad \sum_{n=1}^{\infty} a(n)n^{-s} = \zeta(s) \sum_{n=1}^{\infty} \frac{b(n)}{n} \log \zeta(ns),$$

where $\zeta(s)$ is the Riemann zeta function, if both series converge absolutely. He then considers various estimates involving $a(n)$. We attempt here to generalize (1.3) and thus obtain a fairly general theorem applicable for a wide class of additive functions which include Duncan's $a(n)$.

2. The Theorem. We recall that an arithmetic function h (i.e. a complex-valued function on the positive integers) is said to be additive, provided $h(mn) = h(m) + h(n)$ whenever $(m, n) = 1$. For such a function, we have, obviously, $h(1) = 0$.

THEOREM. *Let h be an additive arithmetic function. For $m, n \geq 1, r \geq 0$, let p_m denote the m -th prime; set $H(m, r) = h(p_m^r)$, and let $E(m, n)$ denote the highest power of p_m dividing n .*

If the double series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} H(m, E(m, n)) n^{-s}$$

is absolutely convergent for $\text{Re } s$ sufficiently large, say for $\text{Re } s > \sigma_0$ then the Dirichlet series $\sum_{n=1}^{\infty} h(n)n^{-s}$ converges to an analytic function $f(s)$ in the half-plane $\text{Re } s > \sigma_0$, and we have the representation

$$f(s) = \zeta(s) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} G(m, r) p_m^{-rs},$$

where $G(m, r) = H(m, r) - H(m, r - 1)$, and $\zeta(s)$ is the Riemann zeta-function.

If, in addition, $H(m, r)$ is independent of m , so that we can write $G(m, r) = g(r) = h(2^r) - h(2^{r-1})$, then in the half-plane $\text{Re } s > \sigma_0$ we have

$$f(s) = \zeta(s) \sum_{n=1}^{\infty} \frac{\log \zeta(ns)}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d),$$

where μ is the Möbius function.

Proof. By virtue of unique factorization of n into primes, we can write for all $n \geq 1$.

$$h(n) = \sum_{m=1}^{\infty} h(p_m^{E(m,n)}),$$

remembering that $h(1) = E(m, 1) = 0$ for all $m \geq 1$. We have also $H(m, 0) = 0$ for all $m \geq 1$, and

$$h(n) = \sum_{m=1}^{\infty} H(m, E(m, n)), \quad n \geq 1.$$

Hence for $s > \sigma_0$ we have, formally:

$$(1) \quad \sum_{n=1}^{\infty} h(n)n^{-s} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} H(m, E(m, n))n^{-s}.$$

We now assume in all that follows that $\text{Re } s > \sigma_0$. Then the absolute convergence of the double series on the right validates all our subsequent steps. We pick up those terms for which m and $E(m, n) = r$ are fixed. These are the terms with

$$n = n_1 \cdot p_m^r, \quad (n_1, p_m) = 1.$$

The contribution of these terms to the right-hand side of (1) is:

$$\begin{aligned} H(m, r) \sum_{n_1=1}^{\infty} n_1^{-s} \cdot p_m^{-rs} &= H(m, r) [\zeta(s) - p_m^{-s} \zeta(s)] p_m^{-rs} \\ &= \zeta(s) H(m, r) [p_m^{-rs} - p_m^{-(r+1)s}]. \end{aligned}$$

Hence we have

$$(2) \quad \sum_{n=1}^{\infty} h(n)n^{-s} = \zeta(s) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} H(m, r) [p_m^{-rs} - p_m^{-(r+1)s}].$$

Applying Abel partial summation to the right side of (2) we obtain

$$(3) \quad \sum_{n=1}^{\infty} h(n)n^{-s} = \zeta(s) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} G(m, r) p_m^{-rs},$$

where $G(m, r) = H(m, r) - H(m, r - 1) = h(p_m^r) - h(p_m^{r-1})$. This is as far as one can go for arbitrary additive functions, and establishes the first part of the theorem.

We now assume that H is independent of its first place. Then so also is G , and we can write $G(m, r) = g(r)$. Hence we have

$$\begin{aligned} \sum_{n=1}^{\infty} h(n)n^{-s} &= \zeta(s) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} g(r)p_m^{-rs} \\ &= \zeta(s) \sum_{r=1}^{\infty} g(r) \sum_{m=1}^{\infty} p_m^{-rs} \\ &= \zeta(s) \sum_{r=1}^{\infty} g(r) \sum_{m=1}^{\infty} \log \zeta(rms) \frac{\mu(m)}{m} \end{aligned}$$

(see, for example, Titchmarsh [6] Chapter 1).

Write the above expression in the right side as a double series in the form

$$\zeta(s) \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} \log \zeta(rms) \frac{\mu(m)}{m} g(r),$$

and then set $rm = n$ and sum on n to obtain, finally:

$$(4) \quad \sum_{n=1}^{\infty} h(n)n^{-s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\log \zeta(ns)}{n} \sum_{d|n} d\mu\left(\frac{n}{d}\right)g(d).$$

This completes the proof of the theorem.

3. Special cases. I. Setting $h(n) = a(n)$ defined in (1.1), we obtain Duncan's formula (1.3). Duncan ([4], [5]) pointed out that some well-known results are immediate consequences of (1.3). The following are two other noteworthy deductions, which the author has not seen mentioned in the literature.

Let $t(n)$ and $t_1(n)$ denote respectively the number of divisors and the number of unitary divisors of n . Setting $G(n) = \log(1+n)$ ($n \geq 0$) and $G(n) = \log 2$ ($n > 0$), $G(0) = 0$ respectively, we obtain

$$(3.1) \quad \sum \frac{\log t(n)}{n^s} = \zeta(s) \sum \frac{1}{n} \log \zeta(ns) \log F(n),$$

where

$$(3.2) \quad \begin{aligned} F(n) &= \prod_{d|n} \left(1 + \frac{1}{d}\right)^{d\mu(n/d)}, \\ \sum \frac{\log t_1(n)}{n^s} &= (\log 2)\zeta(s) \sum \frac{1}{n} \log \zeta(ns)\mu(n). \end{aligned}$$

II. If $G(n)$ is an arbitrary function and

$$V(n) = \sum_{d|n} d(G(d) - G(d - 1))\phi(n/d),$$

$$h(n) = \sum_{i=1}^n \sum_{z=1}^{a_i} \sum_{D|z} (G(D) - G(D - 1)),$$

then $\sum_{n=1}^{\infty} h(n)/n^s = \zeta(s) \sum_{n=1}^{\infty} \log \zeta(ns) V(n)/n$. This is easily deduced from the theorems. In particular, setting $G(n) = \pi(n)$ the number of primes not exceeding n , and $G(n) = \sum_{p \leq n} p$, p being a prime, we obtain respectively,

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{i=1}^r w(1) + w(2) + \dots + w(a_i) \right) / n^s \\ = \zeta(s) \sum_{n=1}^{\infty} \log \zeta(ns) \left(\sum_{p|n} \phi(n/p)(n/p) \right); \end{aligned}$$

and

$$\sum_{n=1}^{\infty} \sum_{i=1}^r (\beta(1) + \beta(2) + \dots + \beta(a_i)) / n^s = \zeta(s) \sum_{n=1}^{\infty} n \log \zeta(ns) \sum_{p|n} \phi(n/p)(p^2/n^2),$$

where $\beta(n)$ is the sum of the distinct prime divisors of n .

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VERY MAGIC SQUARES

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A square matrix A is "magic" in the weakest sense if it has a generalized "doubly stochastic" property, namely when all of its row sums and column sums have the same value s . The matrix is even more "magic" when the sums of its elements along certain diagonals are also equal to s .

If A is an $m \times m$ matrix whose elements are denoted by A_{xy} for $1 \leq x, y \leq m$, let us say a *generalized diagonal* of A is the set of all A_{xy} such that $ax + by \equiv c$ (modulo m), for some given integers a, b, c with a and b relatively prime to each other. For example, a 5×5 matrix has 30 distinct generalized diagonals, namely the sets of elements of the same value in the following squares: