A CLASS OF ADDITIVE FUNCTIONS

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1. Introduction. Throughout this note, let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the representation of $n > 1$ as a product of powers of distinct primes, and define $\Omega_k(n) = a_1^k + \cdots + a_r^k$; $w(n) = \Omega_0(n)$; $\Omega(n) = \Omega_1(n)$. In a series of interesting papers ([1]–[5]) R. L. Duncan considered these functions $\Omega_k(n)$ and in particular obtained identities ([4], Theorem 1) which generalize some results in Titchmarsh ([6] Ch. 1, eqs 1.6.2 and 1.6.3). In the paper [5] to appear, an advance copy of which he kindly sent me, he considers an even more general class of additive functions, given by

$$(1.1) \qquad a(n) = G(a_1) + \cdots + G(a_r) \qquad \text{for } n > 1; \quad a(1) = 0.$$

Here $G(n)$ is an arbitrary arithmetic function for which $G(0) = 0$ and $G(n) \geqq G(n-1)$ for $n \geqq 1$. Duncan establishes the following result: if

$$(1.2) \qquad b(n) = \sum_{d/n} \{G(d) - G(d-1)\} d\mu(n/d),$$

then

$$(1.3) \qquad \sum_{n=1}^{\infty} a(n) n^{-s} = \zeta(s) \sum_{n=1}^{\infty} \frac{b(n)}{n} \log \zeta(ns),$$

where $\zeta(s)$ is the Riemann zeta function, if both series converge absolutely. He then considers various estimates involving $a(n)$. We attempt here to generalize (1.3) and thus obtain a fairly general theorem applicable for a wide class of additive functions which include Duncan's $a(n)$.

2. The Theorem. We recall that an arithmetic function h(i.e. a complex-valued function on the positive integers) is said to be additive, provided $h(mn) = h(m) + h(n)$ whenever $(m, n) = 1$. For such a function, we have, obviously, $h(1) = 0$.

THEOREM. *Let h be an additive arithmetic function. For $m, n \geqq 1$, $r \geqq 0$, let p_m denote the m-th prime; set $H(m, r) = h(p_m^r)$, and let $E(m, n)$ denote the highest power of p_m dividing n.*

If the double series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} H(m, E(m, n)) n^{-s}$$

is absolutely convergent for Re s *sufficiently large, say for* Re $s > \sigma_0$ *then the Dirichlet series $\sum_{n=1}^{\infty} h(n) n^{-s}$ converges to an analytic function $f(s)$ in the half-plane* Re $s > \sigma_0$, *and we have the representation*

$$f(s) = \zeta(s) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} G(m, r) p_m^{-rs},$$

where $G(m, r) = H(m, r) - H(m, r-1)$, *and* $\zeta(s)$ *is the Riemann zeta-function.*

If, in addition, $H(m, r)$ *is independent of* m, *so that we can write* $G(m, r) = g(r) = h(2^r) - h(2^{r-1})$, *then in the half-plane* $\operatorname{Re} s > \sigma_0$ *we have*

$$f(s) = \zeta(s) \sum_{n=1}^{\infty} \frac{\log \zeta(ns)}{n} \sum_{d \mid n} d\mu\left(\frac{n}{d}\right) g(d),$$

where μ *is the Möbius function.*

Proof. By virtue of unique factorization of n into primes, we can write for all $n \geq 1$.

$$h(n) = \sum_{m=1}^{\infty} h(p_m^{E(m,n)}),$$

remembering that $h(1) = E(m, 1) = 0$ for all $m \geqq 1$. We have also $H(m, 0) = 0$ for all $m \geqq 1$, and

$$h(n) = \sum_{m=1}^{\infty} H(m, E(m, n)), \qquad n \geqq 1.$$

Hence for $s > s_0$ we have, formally:

$$\sum_{n=1}^{\infty} h(n) n^{-s} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} H(m, E(m, n)) n^{-s}. \tag{1}$$

We now assume in all that follows that $\operatorname{Re} s > \sigma_0$. Then the absolute convergence of the double series on the right validates all our subsequent steps. We pick up those terms for which m and $E(m, n) = r$ are fixed. These are the terms with

$$n = n_1 \cdot p_m^r, \quad (n_1, p_m) = 1.$$

The contribution of these terms to the right-hand side of (1) is:

$$\begin{aligned} H(m, r) \sum_{n_1=1}^{\infty} n_1^{-s} \cdot p_m^{-rs} &= H(m, r)[\zeta(s) - p_m^{-s}\zeta(s)] p_m^{-rs} \\ &= \zeta(s) H(m, r)[p_m^{-rs} - p_m^{-(r+1)s}]. \end{aligned}$$

Hence we have

$$\sum_{n=1}^{\infty} h(n) n^{-s} = \zeta(s) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} H(m, r)[p_m^{-rs} - p_m^{-(r+1)s}]. \tag{2}$$

Applying Abel partial summation to the right side of (2) we obtain

$$\sum_{n=1}^{\infty} h(n) n^{-s} = \zeta(s) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} G(m, r) p_m^{-rs}, \tag{3}$$

where $G(m, r)=H(m, r)-H(m, r-1)=h(p_m^r)-h(p_m^{r-1})$. This is as far as one can go for arbitrary additive functions, and establishes the first part of the theorem.

We now assume that H is independent of its first place. Then so also is G, and we can write $G(m, r)=g(r)$. Hence we have

$$\begin{aligned}\sum_{n=1}^{\infty} h(n)n^{-s} &= \zeta(s)\sum_{m=1}^{\infty}\sum_{r=1}^{\infty} g(r)p_m^{-rs} \\ &= \zeta(s)\sum_{r=1}^{\infty} g(r)\sum_{m=1}^{\infty} p_m^{-rs} \\ &= \zeta(s)\sum_{r=1}^{\infty} g(r)\sum_{m=1}^{\infty}\log\zeta(rms)\frac{\mu(m)}{m}\end{aligned}$$

(see, for example, Titchmarsh [6] Chapter 1).

Write the above expression in the right side as a double series in the form

$$\zeta(s)\sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\log\zeta(rms)\frac{\mu(m)}{m}g(r),$$

and then set $rm=n$ and sum on n to obtain, finally:

$$\sum_{n=1}^{\infty} h(n)n^{-s} = \zeta(s)\sum_{n=1}^{\infty}\frac{\log\zeta(ns)}{n}\sum_{d\mid n} d\mu\left(\frac{n}{d}\right)g(d). \tag{4}$$

This completes the proof of the theorem.

3. Special cases. I. Setting $h(n)=a(n)$ defined in (1.1), we obtain Duncan's formula (1.3). Duncan ([4], [5]) pointed out that some well-known results are immediate consequences of (1.3). The following are two other noteworthy deductions, which the author has not seen mentioned in the literature.

Let $t(n)$ and $t_1(n)$ denote respectively the number of divisors and the number of unitary divisors of n. Setting $G(n)=\log(1+n)(n\geqq 0)$ and $G(n)=\log 2(n>0)$, $G(0)=0$ respectively, we obtain

$$\sum\frac{\log t(n)}{n^s} = \zeta(s)\sum\frac{1}{n}\log\zeta(ns)\log F(n), \tag{3.1}$$

where

$$F(n) = \prod_{d\mid n}\left(1+\frac{1}{d}\right)^{d\mu(n/d)},$$

$$\sum\frac{\log t_1(n)}{n^s} = (\log 2)\zeta(s)\sum\frac{1}{n}\log\zeta(ns)\mu(n). \tag{3.2}$$

II. If $G(n)$ is an arbitrary function and

$$V(n) = \sum_{d\mid n} d(G(d)-G(d-1))\phi(n/d),$$

$$h(n) = \sum_{i=1}^{n} \sum_{x=1}^{a_i} \sum_{D|x} (G(D) - G(D-1)),$$

then $\sum_{n=1}^{\infty} h(n)/n^s = \zeta(s) \sum_{n=1}^{\infty} \log \zeta(ns) V(n)/n$. This is easily deduced from the theorems. In particular, setting $G(n) = \pi(n)$ the number of primes not exceeding n, and $G(n) = \sum_{p \leq n} p$, p being a prime, we obtain respectively,

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{r} w(1) + w(2) + \cdots + w(a_i)\right) \Big/ n^s = \zeta(s) \sum_{n=1}^{\infty} \log \zeta(ns) \left(\sum_{p|n} \phi(n/p)(n/p) \right);$$

and

$$\sum_{n=1}^{\infty} \sum_{i=1}^{r} (\beta(1) + \beta(2) + \cdots + \beta(a_i)) \Big/ n^s = \zeta(s) \sum_{n=1}^{\infty} n \log \zeta(ns) \sum_{p|n} \phi(n/p)(p^2/n^2),$$

where $\beta(n)$ is the sum of the distinct prime divisors of n.

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VERY MAGIC SQUARES

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A square matrix A is "magic" in the weakest sense if it has a generalized "doubly stochastic" property, namely when all of its row sums and column sums have the same value s. The matrix is even more "magic" when the sums of its elements along certain diagonals are also equal to s.

If A is an $m \times m$ matrix whose elements are denoted by A_{xy} for $1 \leq x, y \leq m$, let us say a *generalized diagonal* of A is the set of all A_{xy} such that $ax + by \equiv c$ (modulo m), for some given integers a, b, c with a and b relatively prime to each other. For example, a 5×5 matrix has 30 distinct generalized diagonals, namely the sets of elements of the same value in the following squares: