$N \to \infty$. Therefore

$$\sum_{-\infty}^{\infty} |c_n d_{-n}| < \infty,$$

and the proof is complete.

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ON RELATIVELY PRIME SEQUENCES

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1. A sequence of integers $\{a_n\}$ $(n \geqq 1)$ will be called relatively prime if $a_n \neq 0$ except for at most one value of n and $(a_m, a_n) = 1$ for all m and n with $m \neq n$. We recall that, if a is a nonzero integer, we define $(a, 0) = |a|$. In this note we consider the following question: Suppose that $\phi(x)$ is a polynomial in x with integral coefficients and $\phi_0(x) = x$, $\phi_{k+1}(x) = \phi(\phi_k(x))$ $(k \geqq 0)$. Suppose also that for all integers x and all $k \geqq 0$ for which $\phi_k(x) \neq 0$ we have $\phi_k(x) | \phi_{k+1}(x)$. We note that this holds if and only if $\phi(0) = 0$, which is therefore assumed throughout. Under what conditions on $\phi(x)$ and for what integral values of x is the sequence of integers $\{f_n(x)\}$ $(n > 0)$ relatively prime, where $f_n(x) = \phi_n(x)/\phi_{n-1}(x)$ $(n = 1, 2, \cdots)$? (For the case $\phi_{n-1}(x) = 0$, see below.) This question arises naturally in view of the known result that the sequence $\{(a^{p^n} - 1)/(a^{p^{n-1}} - 1)\}$ $(n \geqq 1$, p prime, $(a-1, p) = 1)$ is relatively prime, and we can write $a^{p^n} - 1 = \phi_n(a-1)$ with $\phi(x) = (x+1)^p - 1$.

A similar problem for the sequence $\{\phi_n(x)\}$ of iterations of $\phi(x)$ was considered by R. Bellman [1] in trying to generalize the well-known result that the sequence of Fermat numbers $\{F_n\} = \{2^{2^n} + 1\}$ is relatively prime and observing the fact that F_n can be written as $F_n = \phi_n(3)$, where $\phi(x) = (x-1)^2 + 1$. In the sequel $\{\phi_n(x)\}$ and $\{f_n(x)\}$ are sequences as already defined and we write $f(x)$ for $f_1(x)$.

If, for some $n \geqq 0$, $\phi_n(x) = 0$, then $\phi_m(x) = 0$ for all $m \geqq n$. In this case we define

$f_{n+1}(x)$ as $\lim_{y\to x} (\phi_{n+1}(y)/\phi_n(y))$, which $=\phi'(0)$. We also have $f_{m+1}(x)=\phi'(0)$ for all $m\geqq n$.

THEOREM 1. *If, for a given integer x, $\{\phi_n(x)\}$ has a vanishing term, the sequence $\{f_n(x)\}$ is relatively prime if and only if $\phi'(0)=\pm 1$.*

Proof. Let $n=r\geqq 0$ be the smallest integer for which $\phi_n(x)=0$, so that $f_n(x)=\phi'(0)$ for all $n>r$. The necessity of the requirement $\phi'(0)=\pm 1$ for the sequence $\{f_n(x)\}$ to be relatively prime is evident, as also its sufficiency if $r=0$. To show its sufficiency if $r>0$, we observe that $f_r(x)=0$ and $f_n(x)\neq 0$ $(n=1, 2, \cdots, r-1)$. Since $\phi(\phi_{r-1}(x))=\phi_r(x)=0$ it follows that $\phi_{r-1}(x)$ is a nonzero integral root of the polynomial equation $\phi(y)/y=0$ and hence is a divisor of $\phi'(0)$ (which by the sufficiency assumption is ± 1), remembering that $\phi'(0)$ is the constant term of the polynomial $\phi(y)/y$. Also since $\phi_{i-1}(x)\,|\,\phi_i(x)$ $(i=1, 2, \cdots, r-1)$ it follows that if $\phi'(0)=\pm 1$ then $\phi_i(x)=\pm 1$ for $0\leqq i\leqq r-1$, thus completing the proof.

REMARK. It follows from the above proof that if $\phi'(0)\neq 0$, the only possible nonzero integers x for which $\phi_n(x)=0$ for some n are the positive and negative divisors of $\phi'(0)$.

As a very simple illustration, setting $\phi(x)=x(1-x)^2$ we have $\phi'(0)=1$ and $\{f_n(0)\}=\{1, 1, \cdots\}$ and $\{f_n(1)\}=\{0, 1, 1, \cdots\}$ are the only cases of $\{f_n(x)\}$ for which the corresponding $\{\phi_n(x)\}$ has a vanishing term; of course, in both cases $\{f_n(x)\}$ is relatively prime.

On the other hand, for $\phi(x)=x(4-x)$ we have $\phi(4)=\phi_2(2)=0$, and correspondingly, $\{f_n(4)\}=\{0, 4, 4, \cdots\}$, $\{f_4(2)\}=\{2, 0, 4, 4, \cdots\}$ are not relatively prime. Theorem 1 is inapplicable here since $\phi'(0)\neq\pm 1$.

THEOREM 2. *Let $\phi(x)$ satisfy*

$$\phi'(0)\neq 0, \tag{1}$$

$$\text{for any } x\neq 0,\ (x, \phi'(0)) = 1 \text{ implies } (\phi(x)/x, \phi'(0)) = 1. \tag{2}$$

Then for any x for which $(x, \phi'(0))=1$ the sequence $\{f_n(x)\}$ $(n=1, 2, \cdots)$ is relatively prime.

Proof. Assume first that $\phi_r(x)\neq 0$ for $r=0, 1, 2, \cdots$. We use an obvious variation of Bellman's argument in [1]. For any $n>m\geqq 1$ we have

$$\begin{aligned} f_n(x) &= \phi_{n-m}(\phi_m(x))/\phi_{n-m-1}(\phi_m(x)) = f_{n-m}(\phi_m(x)) \\ &\equiv f_{n-m}(0) \pmod{\phi_m(x)} \equiv f_{n-m}(0) \pmod{f_m(x)} \equiv \phi'(0) \pmod{f_m(x)}, \end{aligned}$$

since $\phi(0)=0$ and so $f_k(0)=\phi'(0)$ for all $k>0$. Thus for all $n>m\geqq 1$

$$(f_m(x), f_n(x)) = (f_m(x), \phi'(0)), \tag{3}$$

which $=1$ if $(x, \phi'(0))=1$, on using (2). The theorem now follows in this case.

We now consider the case in which $\{\phi_n(x)\}$ has at least one zero term for $n\geqq 0$.

The case $\phi_0(x)=x=0$ offers no difficulty, for then $f_n(0)=\phi'(0)$ for all $n\geqq 1$.

Also $x=0$ satisfies $(x, \phi'(0))=1$ only if $\phi'(0)=\pm 1$, and the truth of the theorem is obvious. Next, let $n=r>0$ be the smallest value of n for which $\phi_n(x)=0$, so that $x\neq 0$. Since $\phi'(0)\neq 0$ by (1), the only vanishing term of $\{f_n(x)\}$ is $f_r(x)$. To prove the theorem in this case we have only to show, in view of Theorem 1, that if $(x, \phi'(0))=1$ then condition (2) implies that $\phi'(0)=\pm 1$. Assume $(x, \phi'(0))=1$. Using (2) we then have $(\phi(x), \phi'(0))=1$ and a repeated application of the same yields $(\phi_r(x), \phi'(0))=1$, that is, $(0, \phi'(0))=1$ giving $\phi'(0)=\pm 1$ and completing the proof of the theorem.

REMARK. This theorem cannot be deduced from Bellman's Theorem 2 of [1] since our $f_m(x)$ is not the mth iterate of $f(x)$ except, as can easily be shown, in the case when $\phi(x)=ax$, $a\neq 0$.

It can be checked that the conditions of the theorem are satisfied for the following cases:

(A) $\phi(x) = (x+2)^4 - 16$, x odd.

(B) $\phi(x) = (x \pm 1)^p \mp 1$, p odd prime, $(x, p) = 1$.

(C) $\phi(x) = x(ax^k + q^r)^q$, q prime, $(a, q) = 1$, $(x, q) = 1$, k and r integers >0.

The last two are in fact included in the case

(D) $\phi(x) = a_1x + a_2x^2 + \cdots + a_kx^k$, $k > 1$, $a_1 \neq 0$, $a_k \neq 0$, $(a_1, a_k) = 1$,
$a_i \equiv 0 \pmod{a_1}$ $(i = 1, 2, \cdots, k-1)$, $(x, a_1) = 1$.

THEOREM 3. *The sequence* $\{f_n(x)\}$ *is relatively prime for all integral* x *if and only if* $\phi'(0)=\pm 1$.

The "if" part is a consequence of Theorem 2 and the "only if" part follows either from Theorem 1 or by considering the sequence

$$\{f_n(0)\} = \{\phi'(0), \phi'(0), \cdots\}.$$

The corresponding result for the sequence of iterations of a polynomial was obtained in [2].

2. Suppose that $\{f_n(x)\}$ is relatively prime for $x=x_0$, and that an infinite number of terms of the sequence $\neq \pm 1$. Then of course the sequence contains an infinite number of prime divisors. This happens, for example, if $\phi(x)$ satisfies the conditions of Theorem 1 and, in addition, if $|\phi(\pm 1)|>1$, or if $\phi(x)>x$ for $x\geqq x_0$, or if $\phi(x_0)>x_0>1$, or if $\phi(x_0)>x_0$ and $\phi(\pm 1)\neq \pm 1$. The inevitable question arises: Does there exist a $\phi(x)$ and an x_0 such that either all or an infinite number of the numbers $f_n(x_0)$ $(n=1, 2, \cdots)$ are primes, or at least have a bounded number of prime divisors? A similar question arises with regard to the iterations of a polynomial $\phi(x)$ as Bellman remarks in [1]. Even in the special case of Fermat numbers, while work on their primality, mostly computational, is in progress, almost nothing seems to be known so far regarding the boundedness or otherwise of the number of their prime divisors.

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THE MASSERA—SCHAFFER EQUALITY

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J. L. Massera and J. J. Schaffer [Annals of Math., 67 (1958) p. 538] have shown that for any two vectors x and y of a normed linear space

$$\| \|y\|x - \|x\|y\| \max(\|x\|, \|y\|) \leqq 2\|x\| \cdot \|y\| \cdot \|x - y\|. \tag{1}$$

Kirk and Smiley [this MONTHLY, 71 (1964) p. 891] have expressed interest in the conditions under which equality holds.

THEOREM. *Two distinct nonzero vectors x and y of a complex normed linear space satisfy the equality in* (1) *iff x and y span an l_2^1 in the underlying real vector space with*

$$\pm \frac{y - x}{\|y - x\|} \quad \text{and} \quad \pm \frac{x}{\|x\|} \left(\text{or } \pm \frac{y}{\|y\|} \right)$$

as the vertices of the unit parallelogram.

The proof is an immediate consequence of the following lemma.

LEMMA. *If ABC is an isosceles triangle in a two dimensional real normed linear space with $\|AB\| = \|AC\|$ and X any point on side AC, then $\|BX\| \geqq \frac{1}{2}\|BC\|$ with equality holding iff $\|AX\| = \frac{1}{2}\|BC\|$ and the unit circle is a parallelogram.*

Proof. Consider points D and E on sides AC and BC respectively such thnt $\|CD\| = \frac{1}{2}\|BC\|$ and the line DE is parallel to line AB. $\|DE\| = \|DC\| = \frac{1}{2}\|BC\|$. If X is on the closed segment DC, the inequality follows from the triangle inequality with equality possible only if $X \equiv D$. If X is on the closed segment AD, consider the point Y on the segment BE such that the line BX is parallel to the line DY. $\|BX\| \geqq \|DY\| \geqq \|DE\| = \frac{1}{2}\|DC\|$, the middle inequality following from the convexity of spheres. Here again equality holds iff $X \equiv D$.

If $\|BD\| = \|DC\| = \frac{1}{2}\|BC\|$ then B and C together with the reflections of these points in D form the vertices of a parallelogram all of whose boundary points are equidistant from D.

Proof of the theorem. The sufficiency is clear. Suppose then that $\|y\| \geqq \|x\| > 0$ and the vectors are linearly independent. The case of linear dependence leads easily to the condition $x = y$. Application of the lemma to the triangle defined by the null vector and the vectors y and $\|y\|x/\|x\|$ implies that