$$= d\binom{k-1}{r} + (a-d)\sum_{j=1}^{r}\sum_{s=1}^{j}\binom{k-1-j}{r-j}\binom{k-1}{s-1}m^{s-1}$$

$$+ d\sum_{j=1}^{r}\sum_{s=0}^{j}\binom{k-1-j}{r-j}\binom{k}{s}m^{s}$$

$$= (a-d)\sum_{s=1}^{r}\sum_{j=s}^{r}\binom{k-1-j}{r-j}\binom{k-1}{s-1}m^{s-1}$$

$$+ d\sum_{s=0}^{r}\sum_{j=s}^{r}\binom{k-1-j}{r-j}\binom{k}{s}m^{s}$$

$$= (a-d)\sum_{s=1}^{r}\binom{k-1}{s-1}\binom{k-s}{r-s}m^{s-1} + d\sum_{s=0}^{r}\binom{k}{s}\binom{k-s}{r-s}m^{s}$$

$$= (a-d)\binom{k-1}{r-1}\sum_{s=1}^{r}\binom{r-1}{s-1}m^{s-1} + d\binom{k}{r}\sum_{s=0}^{r}\binom{r}{s}m^{s}$$

$$= (a-d)\binom{k-1}{r-1}(m+1)^{r-1} + d\binom{k}{r}(m+1)^{r}$$

for $1 \leqq r \leqq k-1$.

Thus, (6) is true for $t=m+1$ if it is true for $t=m$ and the result follows by mathematical induction.

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SOME REMARKS ON THE PARTITION FUNCTION

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1. Preliminaries. Kolberg [2] was probably the first to prove that the partition function $p(n)$ (which denotes the number of unrestricted partitions of the positive integer n) takes both odd and even values, each of them infinitely often. This result is also contained in the Advanced problem No. 4944 (1961, 76) proposed by M. Newman and solved by J. H. van Lint and the proposer (1962, 69, p. 175) in this Monthly. As a wide generalization of this result M. Newman [3] conjectured that for all integers $m \geqq 2$ each of the congruences

$$p(n) \equiv r \pmod{m}, \qquad 0 \leqq r \leqq m-1,$$

his infinitely many solutions in positive integers n. He proved this conjecture for $m=2, 5, 13, 65$ ([3], [4]).

In this note we prove that $p(2n+1)$ takes even and odd values, each of them infinitely often. It should be noted that our result implies Kolberg's result and yet is not covered by Newman's conjecture. We also prove some other congruences and end up with some conjectures.

2. A recurrence formula with a congruence. In the sequel we write

$$\phi(x) = \prod_{n=1}^{\infty} (1 - x^n), \qquad |x| < 1;$$

$\sigma(n)$ and $t(n)$ for the sum and number of divisors of n respectively, and define $\sigma(0)=0$; $t(0)=p(0)=1$, $p(n)=0$ for $n<0$. Also we write θ for the operator xd/dx and recall that

$$1/\phi(x) = \sum p(n)x^n, \tag{2.1}$$

$$\theta \log(1/\phi(x)) = \sum \sigma(n)x^n, \tag{2.2}$$

where, unless otherwise specified, all summations are from $n=0$ to $n=\infty$.

In addition to the classical result due to Euler on the power series expansion for $\phi(x)$ we use the following well-known and easily proved result:

LEMMA 1. *For all positive integers m and all divisors $d>2$ of* 24,

$$\sigma(dm-1) \equiv 0 \pmod{d}. \tag{2.3}$$

This holds for $d=2$ if and only if $dm-1$ is not a square integer. (Of course (2.3) holds trivially for $d=1$.) A simple proof of this result, given in [**5**], is based on the fact that if $d>2$, the integer $N=dm-1$ is nonsquare, and for any $t|N$, $t+(N/t)\equiv 0 \pmod{d}$. This leads to

LEMMA 2. *Let $d\geqq 2$ be a divisor of* 24. *For all nonsquare N for which*

$$N \equiv -1 \pmod{d} \tag{2.4}$$

we have

$$p(N) \equiv \sum_{k>0} (-1)^{k-1} \left\{\left(1 + \frac{k(3k-1)}{2}\right) p(N - k(3k-1)/2) + \left(1 + \frac{k(3k+1)}{2}\right) p(N - k(3k+1)/2)\right\} \pmod{d}.$$

REMARK. If $d>2$, any N satisfying (2.4) is a nonsquare integer.

Proof. From (2.1) and (2.2) we have

$$\phi(x) \sum np(n)x^n = \sum \sigma(n)x^n.$$

Equating the coefficients of x^N on both sides and using Lemma 1 we have the required result.

COROLLARY. *For N, d as given above,*

(2.5) $$p(N-1)+2p(N-2)-5p(N-5)-7p(N-7)+\cdots\equiv 0 \pmod{d},$$

the general term on the left being $(-1)^{k-1}k(3k\pm1)/2\ p(N-k(3k\pm1)/2)$.

This follows on combining the result of Lemma 2 with Euler's power series expansion for $\phi(x)$. In particular, setting $d=3$ and observing that $k(3k-1)/2\equiv k \pmod 3$ and $k(3k+1)/2\equiv -k \pmod 3$ we have for *all* $N\equiv 2 \pmod 3$,

$$\sum_{k\geqq 1, 3\nmid k}(-1)^{[k/3]}\{p(N-k(3k-1)/2)-p(N-k(3k+1)/2)\}\equiv 0 \pmod 3.$$

3. Parity of $p(2n+1)$.

THEOREM 1. *The congruences* $p(2n+1)\equiv 0 \pmod 2$ *and* $p(2n+1)\equiv 1 \pmod 2$ *have each an infinite number of solutions in n.*

Proof. We apply an argument similar to Kolberg's to the relation

(3.1) $$p(N)+p(N-2)+p(N-12)+p(N-22)+\cdots\equiv 0 \pmod 2,$$

which is valid for any positive odd integer N which is not a perfect square, where in the left member of (3.1) the general term is $p(N-k(3k\pm1)/2)$ provided $k(3k\pm1)/2$ is even. This relation is a consequence of Lemma 2.

Assuming, then, that the theorem is false, let m be the largest odd integer for which $p(m)$ is even (odd). Setting $r=32m+8$ and $N=m+r(3r-1)/2$ it is easily verified that $N\equiv 2 \pmod 3$ and hence not a square integer. Further, since N is odd, the congruence (3.1) applies and the last term in the left member of the congruence is $p(m)$. Setting $a_k=k(3k-1)/2$, $b_k=k(3k+1)/2$, the terms of the sequence $a_1, b_1, a_2, b_2, \cdots$ taken modulo 2 recur, the recurrence cycle being 1, 0, 1, 1, 0, 1, 0, 0. Recalling the values of r and N in terms of m it is seen that in the left member of (3.1) the number of terms is even, being $32m+8$; of these the only even (odd) term is $p(m)$. Hence for this choice of N the congruence (3.1) cannot hold, thus proving the theorem.

4. The function $s_r(n)$. Let $a_1, a_2, \cdots, a_{p(n)}$ denote the number of parts (summands) in the $p(n)$ partitions of n arranged in some order. We define $s_r(n)=a_1^r+\cdots+a_{p(n)}^r$ so that $s_o(n)=p(n)$. We shall write $s(n)$ for $s_1(n)$. Let

$$\phi(x, b)=(1-bx)(1-bx^2)\cdots$$

so that $1/\phi(x, b)=1+B_1x+B_2x^2+\cdots$, where $B_n=b^{a_1}+\cdots+b^{a_{p(n)}}$. It is easily seen, using differentiation with respect to b a couple of times and setting $b=1$ (and defining $s_r(n)=0$ for $n\leqq 0$) that

(4.1) $$\begin{aligned}\phi(x)\sum s(n)x^n &= \sum t(n)x^n,\\ \phi(x)\sum s_2(n)x^n &= \left(\sum t(n)x^n\right)^2+\sum \sigma(n)x^n.\end{aligned}$$

Using the Euler expansion for $\phi(x)$, the congruence properties for $\sigma(n)$ in Lemma

1 and the fact that $t(n) \equiv 0 \pmod 2$ unless n is a square we can prove not only recurrence relations for $s(n)$ and $s_2(n)$ as in Lemma 2, but also:

THEOREM 2. *The functions $s(n)$ and $s_2(n)$ take infinitely often even as well as odd values.*

The proof of this is analogous to that of Theorem 1. The function $s(n)$ occurs in a recent paper of Fine [1], where (4.1) is obtained by a different method. A generating series for $s_r(n)$ for general values of r seems to be fairly complicated.

5. Concluding remarks. The author has been unable to prove the analogue of Theorem 1 for $p(2n)$—namely that, for infinitely many n, $p(2n)$ is odd (even), but believes it to be true. It might be conjectured that $s_r(n)$ takes both even and odd values, each infinitely often, for all integers $r \geqq 0$, and probably more, that for all integers $k \geqq 1$ and all a, $0 \leqq a < k$, each of the congruences $s_r(nk+a) \equiv 0 \pmod k$, $s_r(nk+1) \equiv 1 \pmod k$ hold for an infinite number of integers n and all integers $r \geqq 0$.

In a recent private communication to the author, Professor O. Kolberg proved that $p(2n)$ takes even as well as odd values infinitely often. Following Kolberg's method, the author has now proved similar results concerning $p(4n+r)$ $(r=0, 1, 2, 3)$. These, as well as further extensions, will appear elsewhere.

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SOME THEOREMS ON EXPANSIVE HOMEOMORPHISMS

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In this note, a new theorem concerning expansive homeomorphisms will be proved, and an elementary proof of a known theorem appears as a corollary to the new theorem mentioned above.

The following theorem was first proved by Jakobsen and Utz in [4]. We now give a new proof.

THEOREM 1. *If D is the closed unit disk, then there does not exist an expansive homeomorphism on D.*

Proof. Let f be an expansive homeomorphism on D with expansive constant δ. Let C be the boundary of D. Since f maps C onto itself, f is expansive on C.