THE BRAUER-RADEMACHER IDENTITY

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1. Introduction. Let n and r be integers, $r>0$ and let $\phi(r)$ and $\mu(r)$ denote the well known arithmetic functions of Euler and Möbius. The so called Brauer-Rademacher identity [1] says that

$$\phi(r) \sum_{\substack{d \mid r \\ (d,n)=1}} d\,\mu\left(\frac{r}{d}\right) \Big/ \phi(d) = \mu(r) \sum_{d \mid (r,n)} d\,\mu\left(\frac{r}{d}\right). \tag{1}$$

Eckford Cohen gave new proofs of this identity in [2], [3] and [4] with some generalizations, in two of which ([2], [3]) he used the principle of inversion for even arithmetic functions, and in the other, the theory of expansion of Cauchy products of such functions. For another proof of (1) and generalization we refer to McCarthy [6].

Now it is well known that

$$\sum_{d \mid (r,n)} d\mu(r/d) = \phi(r)\mu(r/g)/\phi(r/g), \qquad g = (r, n)$$

and in fact the expressions on both sides of the equation are identical with the Ramanujan sum $c(n, r) = \sum \exp(2\pi i n x/r)$, summed for all x varying over a reduced residue system modulo r. Hence (1) becomes

$$\sum_{\substack{d \mid r \\ (d,n)=1}} d\mu(r/d)/\phi(d) = \mu(r)\mu(r/g)/\phi(r/g). \tag{2}$$

It is important to note that the arithmetic function $f(r) = r/\phi(r)$ is multiplicative and has the additional property that, for every prime p,

$$f(p) = f(p^2) = f(p^3) = \cdots. \tag{3}$$

We wish to show that every such multiplicative function satisfies an identity of the Brauer-Rademacher type. In deriving this identity we do not use the methods of Cohen or McCarthy, but simple properties of multiplicative arithmetic functions of two arguments.

2. Preliminaries. All functions considered in the sequel are assumed to be complex-valued arithmetic functions. The letter p always denotes a prime.

A function $f(r)$ is said to be a multiplicative function of r (or, briefly, multiplicative) if $f(1) = 1$ and $f(r_1 r_2) = f(r_1)f(r_2)$ whenever $(r_1, r_2) = 1$. This definition can be extended for functions of more than one argument. Thus the function $f(r, s)$ is said to be a multiplicative function of both arguments r and s if $f(1, 1) = 1$ and

$$f(r_1 r_2, s_1 s_2) = f(r_1, s_1) f(r_2, s_2)$$

whenever $(r_1s_1, r_2s_2)=1$. The function $f(r, s)$ is said to be multiplicative in r if, by keeping s arbitrarily fixed, $f(r, s)$ is a multiplicative function of r. A similar definition holds for $f(r, s)$ to be multiplicative in s.

If $f(r, s)$ is a multiplicative function of the arguments r and s, it need not be multiplicative in r or s, but $f(r, 1)$ and $f(1, s)$ are multiplicative in r and s respectively. It has also to be noted that a multiplicative function $f(r, s)$ is uniquely determined whenever, for every prime p and all nonnegative integers a and b, the values of $f(p^a, p^b)$ are known.

The following result is useful:

LEMMA. *If $f(r)$ and $g(r)$ are multiplicative and*

$$h(n, r) = \sum_{\substack{d \mid r \\ (d,n)=1}} f(d)g(r/d),$$

then $h(n, r)$ is a multiplicative function of both arguments and is also multiplicative in r.

Proof. If $r=r_1r_2$, $n=n_1n_2$ and $(r_1n_1, r_2n_2)=1$, then any divisor d of r for which $(d, n_1n_2)=1$ can be uniquely written as $d=d_1d_2$ where

$$d_1 \mid r_1,\ d_2 \mid r_2,\ (d_1, n_1) = (d_2, n_2) = 1. \tag{4}$$

Conversely, whenever d_1 and d_2 satisfy (4), $d_1d_2 \mid r$ and $(d_1d_2, n)=1$. Thus

$$h(n, r) = \sum_{\substack{d_1 \mid r_1 \\ (d_1,n_1)=1}} f(d_1)g(r_1/d_1) \cdot \sum_{\substack{d_2 \mid r_2 \\ (d_2,n_2)=1}} f(d_2)g(r_2/d_2) = h(n_1, r_1)h(n_2, r_2),$$

showing that $h(n, r)$ is multiplicative in its two arguments.

A similar proof can be given to show that $h(n, r)$ is multiplicative in r.

We note that $h(n, r)$ need not be multiplicative in n.

3. THEOREM. *Let $f(r)$ be a multiplicative function such that $f(p^a)=f(p^{a+1})$ for all $a \geqq 1$ and every prime p. Let $h(r)$ be any multiplicative function for which $h(p) = f(p)-1$ for every prime p. Then for $r \geqq 1$, $n \geqq 1$,*

$$\sum_{\substack{d \mid r \\ (d,n)=1}} f(d)\mu(r/d) = \mu(r)\mu(r/t)h(r/t), \tag{5}$$

where $t=(n, r)$.

Proof. It follows from the lemma that the left side of (5) is a multiplicative function of both the arguments r and n; also it is easily verified that the same result holds for the right side of (5). Thus the theorem follows if it is shown that (5) holds for $r=p^a$, $n=p^b$ for every prime p and all nonnegative integers a and b. The result is trivially true for $a=b=0$. Denoting the left and right sides of (5) by $F(n, r)$ and $H(n, r)$ respectively, we have for the case $b>0$,

$$F(p^b, p^a) = \sum_{\substack{0 \leq k \leq a \\ \min(k,b)=0}} f(p^k)\mu(p^{a-k}) = f(1)\mu(p^a) = \mu(p^a).$$

If $b=0$ then

$$F(p^b, p^a) = \sum_{d|p^a} f(d)\mu(p^a/d); \tag{6}$$

for $a=1$ this becomes $\mu(1)f(p)+\mu(p)f(1)=f(p)-1$, while for $a\geqq 2$ (6) gives $\mu(1)f(p^a)+\mu(p)f(p^{a-1})=f(p^a)-f(p^{a-1})=0$. Thus

$$F(p^b, p^a) = \begin{cases} 1, & a=b=0 \\ f(p)-1, & a=1, b=0 \\ 0, & a \geqq 2, b=0 \\ \mu(p^a) & b>0. \end{cases} \tag{7}$$

Similarly, it can be verified directly that the function $H(p^b, p^a) =\mu(r)\mu(r/t)h(r/t)$ has the same values as on the right side of (7) for different values of a and b. This completes the proof of the theorem.

REMARK. In the statement of the theorem, for a given $f(r)$, we can take for $h(r)$ any multiplicative function subject to the only restriction $h(p)=f(p)-1$ for every prime p. Thus the values of $h(p^a)$ for $a>1$ can be arbitrarily chosen and therefore $h(r)$ can be chosen in an infinite number of ways.

4. Applications. (1) Let $f(r)=1$ for all r and $h(r)=[1/r]=1$ or 0 according as $r=1$ or >1. The theorem gives

$$\sum_{\substack{d|r \\ (d,n)=1}} \mu(r/d) = [1/r].$$

This result, of course, can easily be seen otherwise. (2) Let $f(r)=2^{w(r)}$, where $w(r)$ is the number of distinct prime divisors of r. Then $f(p^k)=2$ for all $k>0$. We should then have $h(p)=1$ and thus we can define either $h(r)=1$ for all r, or $h(r)=|\mu(r)|$. The corresponding identities will be

$$\sum_{\substack{d|r \\ (d,n)=1}} 2^{w(d)}\mu(r/d) = \mu(r)\mu(r/t) = \mu(r)\mu(r/t)\,|\,\mu(r/t)\,|\,.$$

(3) Taking $f(r)=r/\phi(r)$ so that $f(p^k)=p/(p-1)$, for all $k>0$, we can choose $h(r)=1/\phi(r)$ since $h(p)=1/(p-1)=f(p)-1$. This gives the Brauer-Rademacher identity.

We can, in fact, get a generalization of this identity by setting

$$f(r) = (r/\phi(r))^a,$$

where a is any real number. In particular, for $a=-1$ we should have $h(p) = -1/p$ so that we can choose, for example,

$$h(r) = \mu(r)/r \quad \text{or} \quad h(r) = \lambda(r)/r;$$

here $\lambda(r)$ is Liouville's function given by $\lambda(r) = (-1)^{\alpha(r)}$, where $\alpha(r)$ denotes the number of prime divisors of r, the multiplicity of each such divisor being taken into account. Thus we obtain

$$\sum_{\substack{d \mid r \\ (d,n)=1}} \phi(d)\mu(r/d)/d = \mu(r)t/r = t\mu(r)\mu(r/t)\lambda(r/t)/r.$$

(4) Let $\phi_k(r)$ denote Jordan's totient function representing the number of ordered sets $\{a_1, \cdots, a_k\}$, each a_i being chosen from a complete residue system mod r in such a way that the g.c.d. of $r, a_1, \cdots, a_k$ is unity.

Choosing $f(r) = \phi_k(r)/r^k$ and $f(r) = r^k/\phi_k(r)$ we can take either $h(r) = \mu(r)/r^k$ or $h(r) = \lambda(r)/r^k$ for the first choice of $f(r)$ and $h(r) = 1/\phi_k(r)$ for the second. In the latter case we get

$$\sum_{\substack{d \mid r \\ (d,n)=1}} d^k\mu(r/d)/\phi_k(d) = \mu(r)\mu(r/t)/\phi_k(r/t).$$

This generalization of the Brauer-Rademacher identity has been obtained by Cohen and is given, in a slightly different form, in formula (3.4) of his paper [5].

Research done under NSF Grant GP1222.

References

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MATHEMATICAL SWIFTIES

"As $x \to 0$, the function $(1/x)(\sin 1/x)$ oscillates," Tom said wildly.

"The function takes on only the values 0 and 1," said Tom with characteristic modesty.

"Rearrangement of this series does not affect its convergence," Tom said unconditionally.

"It's easy to construct the reals, given the rationals," Tom said cuttingly.