A GENERATING FUNCTION FOR A CLASS OF ARITHMETIC FUNCTIONS

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We propose to generalize here the results given by V. C. Harris and L. T. Warren [1] concerning the function $\sigma_k(n)$ which stands for the sum of the kth powers of the divisors of n.

Let $g(n)$ be any multiplicative function of n, so that we have $g(mn) = g(m)g(n)$ whenever m and n are mutually prime. Let $h(n) = \sum g(d)$, summed over the divisors of n, so that $h(n)$ is also a multiplicative function of n. Let a be any positive integer and r the largest divisor of a which is prime to n. Let $a = rs$. Then, as a generalization of Theorem 1 of [1], we have

THEOREM 1. *Let $f(a, n)$ be the arithmetic function defined by the relation*

$$\sum_{n=1}^{\infty} \frac{f(a, n)x^n}{1 - x^n} = \sum_{n=1}^{\infty} h(an)x^n.$$

Then $f(a, n) = h(r)g(sn)$.

Proof. We have $h(an) = \sum_{d|n} f(a, d)$, so that

$$f(a, n) = \sum_{d|n} h(an/d)\mu(d) = \sum_{d|ns} h(an/d)\mu(d) = h(r)\sum_{d|ns} h(sn/d)\mu(d), \tag{1.1}$$

$\mu(n)$ being the Möbius function. Also $h(n) = \sum_{d|n} g(d)$, so that by the Möbius Inversion formula,

$$g(ns) = \sum_{d|ns} h(sn/d)\mu(d).$$

The theorem now follows on combining this result with (1.1).

THEOREM 2. *Let $g(n)$ be a positive valued and unconditionally multiplicative function of n, so that $g(mn) = g(m)g(n)$ for all positive integers m and n. Then the function $k(a, n)$ given by $k(a, n) = f(a, n)/g(n)$ is periodic in n with least period P, where P is the product of the distinct prime factors of a.*

Proof. If b is any factor of a such that $(b, n) = 1$, then $(b, n+P) = 1$ and conversely. Hence r and s are unaltered by replacing n by $n+P$. Now

$$\begin{aligned} f(a, n)/g(n) &= h(r)g(sn)/g(n) \\ &= h(r)g(s)g(n)/g(n) = h(r)g(s). \end{aligned}$$

Hence $k(a, n)$ has period P in n.

We can easily show that P is the least period. For if R is the least period we have $k(a, n) = k(a, n+R)$ for all n. Taking $n = a$ we get $h(1)g(a) = h(t)g(u)$ where t is the largest factor of a such that $(t, a+R) = 1$ and $a = tu$. Since $g(n)$ is positive and unconditionally multiplicative it follows that $h(n)$ is positive and multiplicative in n, and in particular $h(1)=1$, since $g(1)=1$. Thus $h(t)g(u)=g(a)=g(ut)=g(u)g(t)$, giving $h(t)=g(t)$. Since $h(t)=\sum_{d|t} g(d)$ it follows that $t=1$. This shows that every prime factor of a is a prime factor of R, proving that P is the least period of $k(a, n)$.

Remarks. A large number of arithmetic functions are of the form $h(n)$. Thus taking $g(n)=1$, $h(n)$ becomes $\tau(n)$, the number of divisors of n. If $g(n)=n^r$ we get $h(n)=\sigma_r(n)$, and thus obtain the results of [1].

Reference

1. V. C. Harris and L. J. Warren, A generating function for $\sigma_r(n)$, this MONTHLY, 66 (1959) 467–472.

MATHEMATICAL NOTES

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ON SIMULTANEOUS HERMITIAN CONGRUENCE TRANSFORMATIONS OF MATRICES

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We shall establish here a few results regarding simultaneous transformation of two matrices. The transpose and conjugate transpose of a square matrix P will be denoted respectively by P' and P^*; similarly for a vector; and $|P|$ will denote the determinant of P. We shall use the notation

$$\begin{bmatrix} P & \\ & Q \end{bmatrix}$$

to denote the direct sum of two square matrices P and Q. A triangular matrix is one in which the elements below the main diagonal are 0.

Our first result is given in the following theorem.

THEOREM 1. *If A and B are square matrices of the same size and are such that for no column vector ξ with complex elements $\xi^*A\xi=\xi^*B\xi=0$, then there exists a nonsingular matrix C such that C^*AC and C^*BC are both triangular matrices.*

Proof. Let A and B be $n \times n$ matrices. Let λ be a root of the equation $|A-\lambda B|=0$. Take a nonnull vector ξ_1 written as a column vector, such that $A\xi_1=\lambda B\xi_1$. Choose a set of linearly independent column vectors $\xi_2, \xi_3, \cdots, \xi_n$ satisfying $\xi_i^*A\xi_1=0$ or $\xi_i^*B\xi_1=0$, $i=2, 3, \cdots, n$ according as $\lambda\neq 0$ or $\lambda=0$.